# 1 3+1 split of spacetime

"Standard" numerical methods for dynamical systems are designed for PDEs that look [roughly] like:

$$\underset{\text{Time derivative (first order in time).}}{\partial_t U} = \underset{\text{Spatial derivatives.}}{A^p \partial_p U} + S \tag{1}$$

But GR looks like: $G_{ab} = 8\pi T_{ab}$ Everything mixed up. <u>But</u> geometrical.

> → Let us <u>locally</u> introduce a time coordinate $t$, assuming now that we have a spacetime $(M, g)$ [not necessarily Einsteinian]. Then we will see how to "split up" the spacetime into space and time geometrically. But throughout remember that $t$ is arbitrary. In the end, we want a form like (1) → <u>Cauchy</u> or <u>"initial value problem"</u>.

Try to express geometry in terms of "intrinsic" and "extrinsic" quantities to $\Sigma_t$.

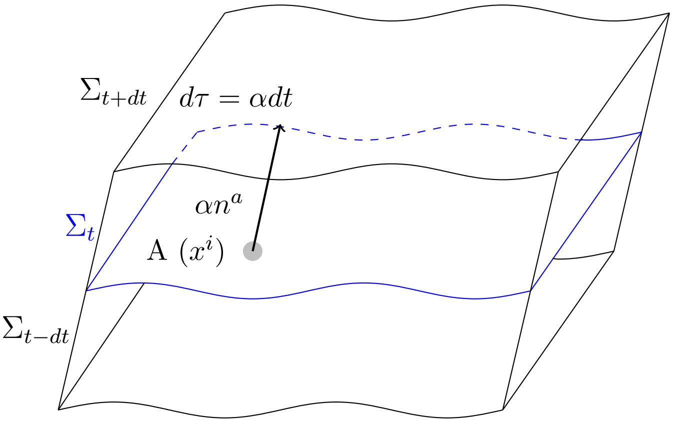


Figure 1: $\Sigma_t$ = level set $t = const$ of $t$.

<u>Notation:</u>

- $a,\ b,\ c,$ ... abstract indices.
- $\mu,\ \nu,\ \delta,$ ... 4D component indices (0,1,2,3).
- $i,\ j,\ k,$ ... spatial indices (1,2,3).
- ( ) symmetrization.
- [ ] antisymmetrization. Like normal.
- (-,+,+,+) signature.
- $g_{ab}$ - 4 metric. $\nabla_a$ - spacetime covariant derivative compatible with $g_{ab}$.
- $\gamma_{ab}$ - 3 metric. $D_a$ - spatial covariant derivative.
- ${}^{(4)}R_{abcd}$ - 4 Riemann tensor, $2\nabla_{[a}\nabla_{b]}V_c =^{(4)} R_{abc}{}^d V_d$.
- $R_{abcd}$ - 3 Riemann tensor.

Lapse function ($\alpha$): $\alpha^{-2} = -(\nabla_a t)(\nabla^a t)$ Lapse → proper time elapsed between hypersurfaces as seen by an observer moving along the normal direction ($d\tau = \alpha dt$). So what is $\alpha^{-2}$?

Remember: $g^{\mu\nu} = g^{ab}(e^\mu)_a(e^\nu)_b = g^{ab}\nabla_a x^\mu \nabla_b x^\nu$ → in coordinate basis.

Unit normal vector: $n^a = -\alpha\,\nabla^a t$ Normal to what?

Consider a curve $X(S) : \mathbb{R} \to M$ with $t = \mathit{constant}$ along $X$.
($M \to \mathbb{R}^4$) (Stick coordinates on). (Take $t$ as time coordinate).
Then $X : X^\mu(S)$ has tangent vector $S^\mu = \frac{d}{dS}X^\mu(S) = (0, \dot{X}^i(S))^T$.
Compute $g_{\mu\nu}n^\mu S^\nu = -\alpha\nabla_\mu t\, S^\mu = -\alpha[1\cdot 0 + 0_i\dot{X}^i(S)] = 0$.
So $n^a$ is normal to the tangent vector of any curve contained in $\Sigma_t$. If $\Sigma_t$ spacelike, $n^a n_a = -1$.

Example: Minkowski: $t$ normal time coordinate

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{aligned} \nabla_\mu t &= (1,0,0,0) \\ \alpha^{-2} &= -\eta^{\mu\nu}(\nabla_\mu t)(\nabla_\nu t) = 1 \\ n^\mu &= -\alpha\nabla^\mu t = (1,0,0,0)^T \end{aligned}$$

The spatial metric: [Sometimes called 3-metric]. $\gamma_{ab} = g_{ab} + n_a n_b$. Check: $n^a\gamma_{ab} = n_b + (n_a n^a)n_b = n_b - n_b = 0$

Example: Minkowski: constant $t$ slices $n^\mu = (1,0,0,0)^T$ and $\gamma_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Projection operator: $\gamma^a{}_b = g^a{}_b + n^a n_b$. Check: $\gamma^a{}_b\gamma^b{}_c = \gamma^a{}_c$ “Repeated application does nothing new”. Good projector operator.

Now we want to decompose tensors using $n^a$ and $\gamma^a{}_b$.

Example: 3+1 split of a vector:

$$V^a = g^a{}_b V^b = -n^a \underbrace{(n_b V^b)}_{\text{Normal component.}} + \underbrace{\gamma^a{}_b V^b}_{\text{Spatial part of } V.}$$

BREAK

Let us now take arbitrary coordinates $x^i$ on the slice $\Sigma_t$. [Ok, in a neighbourhood, but could make argument purely in submanifold.]
We have: $\nabla_\mu t = (1,0,0,0) \quad\to\quad n_\mu = (-\alpha, \vec{0})$.
We also need the partial time derivative “time vector” $t^\mu = (1, \vec{0})^T$ in our coordinates. In

Minkowski (in standard coordinates) $n^\mu = (1,\vec{0})^T = (\partial_t)^\mu$. But $n^\mu \neq t^\mu$ in general.

$$n^\mu = n^a \nabla_a x^\mu = \left(\alpha^{-1}, -\tfrac{1}{\alpha}(-\alpha n^i)\right)^T = \left(\alpha^{-1}, -\alpha^{-1}(-\alpha n^i)\right)^T$$

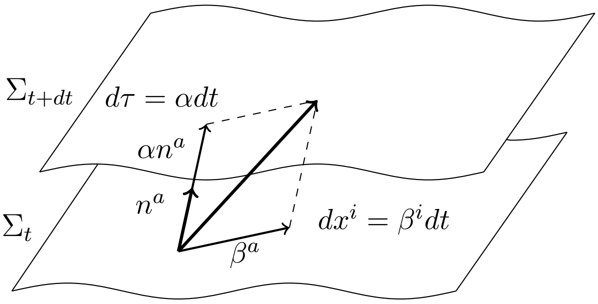
$$\rightarrow t^\mu \equiv (\partial_t)^\mu = (1,\vec{0})^T = \alpha n^\mu + \underbrace{(\alpha n)^i \delta_i{}^\mu} = \alpha n^\mu + \beta^\mu$$

$\beta^\mu$ components of “shift vector”.

Shift vector is spatial: $\beta^\mu n_\mu = 0$

Abstract indices: $t^a = \alpha n^a + \beta^a \qquad \beta^a n_a = 0$

Recall: (Coordinate) basis one-forms: $\nabla_a X^\mu$ .
(Coordinate) basis vectors: $\left(\frac{\partial}{\partial X^\mu}\right)^a$ .
These satisfy (by definition): $\left(\frac{\partial}{\partial X^\mu}\right)^a \nabla_a X^\nu = \delta_\mu{}^\nu$
So simply we have: $\nabla_a t$ → time coordinate basis one-form, $t^a$ → time coordinate basis vector. (Weakness in NR textbooks.)



Easy exercise - show:

$$g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \frac{\beta^i}{\alpha^2} \\ \frac{\beta^j}{\alpha^2} & \gamma^{ij} - \frac{\beta^i\beta^j}{\alpha^2} \end{pmatrix}. \qquad \text{It follows:} \qquad g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}.$$

Another exercise: Check $g^{\mu\nu} g_{\nu\delta} = \delta^\mu_\delta$ .

Out of curiosity: $\gamma^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{ij} \end{pmatrix}$ and $\gamma_{\mu\nu} = \begin{pmatrix} \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}$.

Now that we have a metric in $\Sigma_t$, covariant derivative of $\gamma_{ab}$?
$D_a X_{b\ldots}{}^{c\ldots} \equiv \gamma_a{}^d \gamma_b{}^e \gamma_f{}^c \ldots \nabla_d X_{e\ldots}{}^{f\ldots}$ with $X$ spatial, i.e. $n \cdot X = 0$ [Contraction on any index].
Linear ✓ Leibniz ✓
Compatibility: $D_a \gamma_{bc} = \gamma_a{}^d \gamma_b{}^e \gamma_c{}^f \nabla_d (g_{ef} + n_e n_f) = \gamma_b{}^e n_e \gamma_a{}^d \gamma_c{}^f \nabla_d n_f + (b \leftrightarrow c) = 0$ as $\gamma_b{}^e n_e = 0$ by construction.

[[→ Might be worried about dimensionality, # components of 4-Christoffels. Inverse spatial metric? This will all work out!]]

Now we've seen how to express one part of $\nabla X$ under the 3+1 decomposition. This was the part "intrinsic" to the slice.

Extrinsic curvature:

Consider 2 spacelike vectors $U^a, V^a$ and take:

$$\begin{aligned} U^a \nabla_a V^b &= U^c {\gamma_c}^a [\nabla_a V^d][{\gamma_d}^b - n_d n^b] \\ &= \underbrace{U^a D_a V^b}_{\text{Covariant derivative in } \Sigma_t.} - \underbrace{U^a {\gamma_a}^c (\nabla_c V^d) n_d n^b}_{\text{Bit "outside".}} = U^a D_a V^b + \underbrace{U^a {\gamma_a}^c (\nabla_c n_d) V^d n^b}_{\text{Note: no } \nabla V \text{ here.}} \qquad \text{[Why?]} \end{aligned}$$

Define "extrinsic curvature":

$$\begin{aligned} K_{ab} &= -{\gamma_a}^c \nabla_c n_b = -{\gamma_a}^c {\gamma_b}^d \nabla_c n_d \ \text{[Why?]} = {\gamma_a}^c {\gamma_b}^d [\nabla_c (\alpha \nabla_d t)] = {\gamma_a}^c {\gamma_b}^d [\alpha \nabla_c \nabla_d t + \nabla_d t \nabla_c \alpha] \\ &= {\gamma_a}^c {\gamma_b}^d [\alpha \nabla_c \nabla_d t - \alpha^{-1} n_d \nabla_c \alpha] = \alpha {\gamma_a}^c {\gamma_b}^d \nabla_c \nabla_d t \quad \text{Symmetric!} \end{aligned}$$

Don't be confused by name. In coordinates: really part of the Christoffels.

So: $\qquad U^a \nabla_a V^b = U^a D_a V^b - (K_{ac} U^a V^c) n^b$ .

Equivalent expressions: $\qquad K_{ab} = -\frac{1}{2} \mathcal{L}_n \gamma_{ab} = -\frac{1}{2} n^c \nabla_c \gamma_{ab} - \gamma_{c(a} \nabla_{b)} n^c = -\nabla_a n_b - n_a a_b$

with $a_b \equiv n^c \nabla_c n_b$ "acceleration of Eulerian observers".

Looks strange. Just check by brute force.

Examples:

- Minkowski, global inertial frame: $n_\mu = (1, 0, 0, 0)$, $K_{\mu\nu} = 0$.
- Schwarzschild spacetime, Schw. coords: $ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$
  $\nabla_\mu t = (1, 0, 0, 0)$; $\alpha = \left(1 - \frac{2M}{r}\right)^{1/2}$; $n_\mu = -\left(1 - \frac{2M}{r}\right)^{1/2} (1, 0, 0, 0)$; $K_{\mu\nu} = 0$.
- Schwarzschild spacetime, Kerr-Schild coordinates:
  New time coordinate: $T = t + 2M \ln \left| \frac{r}{2M} - 1 \right|$,
  so the metric is: $ds^2 = -\left(1 - \frac{2M}{r}\right) dT^2 + \frac{4M}{r} dT\, dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\Omega^2$
  → No coordinate singularity at the horizon.
  $\alpha = \left(1 + \frac{2M}{r}\right)^{-1/2}$, $K_{rr} = -\frac{2M(M+r)}{\sqrt{r^6(2M+r)}}$.

So far, introduced $t$. Derived / defined: $\alpha$ (some component of metric), $n_a$, $\gamma_{ab}$ (some other components. Which ones?), $\beta^a$. → Bits of the metric.

Then we had: ${\gamma_a}^b \nabla_b \{\text{spatial tensor}\} \rightarrow \begin{cases} K_{ab} - \text{Extrinsic curvature } \text{{\scriptsize "How slice is curved in ambient space".}} \\ D_a - \text{intrinsic covariant derivative} \end{cases}$

For $n^a \nabla_a \{\text{spatial tensor}\}$ → Typically introduce $\mathcal{L}_n$"tensor" (+ extrinsic curvature terms).

Example decomposition of the spacetime covariant derivative of a vector:

$$\begin{aligned} &\nabla_a V^b = {g_a}^c \nabla_c V^b = ({\gamma_a}^c - n_a n^c) \nabla_c V^b = {\gamma_a}^c \nabla_c V^b - n_a n^c \nabla_c V^b = {\gamma_a}^c \nabla_c V^d [{g_d}^b] - n_a (n^c \nabla_c V^b) = \\ &{\gamma_a}^c \nabla_c V^d [{\gamma_d}^b - n_d n^b] - n_a (\mathcal{L}_n V^b + V^c \nabla_c n^b) = {\gamma_a}^c {\gamma_d}^b \nabla_c V^d - n^b {\gamma_a}^c n_d \nabla_c V^d - n_a \mathcal{L}_n V^b - n_a V^c \nabla_c n^b \\ &= D_a V^b + n^b V^d {\gamma_a}^c \nabla_c n_d - n_a \mathcal{L}_n V^b - n_a V^d {\gamma_d}^c \nabla_c n^b = D_a V^b - n^b K_{ac} V^c - n_a \mathcal{L}_n V^b + n_a {K_c}^b V^c \end{aligned}$$

Next time: Check counting and break up curvature.

BREAK

[Quick recap]

[coordinates] Given / choose $t, x^i$ [Drawing of $\Sigma_t$'s embedding.]
→ [metric] $\alpha$ - lapse, normal vector $n^a$; $\beta^a$ - shift vector; $\gamma_{ab}$ - spatial metric / projector operator
→ [Christoffels] $D_a$ - intrinsic covariant derivative; $K_{ab}$ - extrinsic curvature "How $\Sigma_t$ is curved by ambient spacetime". → [curvature??]

Want to 3+1 decompose curvature.

First: spring cleaning and counting

We've introduced several spatial tensors $\alpha, \gamma_{ab}, K_{ab}, \beta^a$.
Spatial should somehow mean that they act on (co)vectors in the (dual) tangent space of points in $\Sigma_t$, $T_p\Sigma_t$.
This is 3-dimensional - so expect to have only 3 value indices. But so far we're stuck with spacetime indices. Let's fix this!

Start with some spatial vector $V^a$. In our coordinates we have: [Drawing of $\Sigma_t$'s embedding.]
$n_\mu = (-\alpha, \vec{0})$, $n^\mu = (\frac{1}{\alpha}, \frac{-\beta^i}{\alpha})^T$ and $V_\mu = (V_0, V_i)$, $V^\mu = (V^0, V^i)^T$.

$$\begin{aligned} V_\mu n^\mu = 0 &\rightarrow V_0 \cdot \tfrac{1}{\alpha} - \tfrac{\beta^i}{\alpha} \cdot V_i = 0 \rightarrow V_0 = \beta^i V_i \\ V^\mu n_\mu = 0 &\rightarrow -\alpha V^0 = 0 \rightarrow V^0 = 0 \end{aligned}$$

→ Similar for other spatial tensors: If you know the spatial $(i, j, k)$ components, you can construct the rest by "$n \cdot tensor$" $= 0$.
Upstairs "0" components will be 0. Downstairs "0" components will pick up terms like $\beta^i X_i$.

Some counting:
Christoffel symbols ${}^{(4)}\Gamma^\mu{}_{\nu\delta}$ - 40 components. Check we have everything:

- $(D_a \rightarrow)$ $\Gamma^i{}_{jk}$ → 18 components
- $K_{ij}$ → +6 components = 24 components so far
- $\partial_\mu \alpha, \partial_\mu \beta^i$ → +16 components = 40 components → So we have it all!

Decomposition of the curvature:
Want to express ${}^{(4)}R_{abcd}$ in terms of $R_{ijkl}, R_{ij}, K_{ij}, ...$
Due to Riemann symmetries, only 3 combinations are non-vanishing.

- Total projection onto the spacelike hypersurface $\Sigma_t$ - the Gauss equation:

$$\gamma_a^e \gamma_b^f \gamma_c^g \gamma_d^h \, {}^{(4)}R_{efgh} = R_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc} = R_{abcd} + 2 K_{a[c} K_{d]b}.$$

- First contract once with $n$, then project - the Codazzi-Mainardi equation:

$$\gamma^e_a\gamma^f_b\gamma^g_c n^d\,{}^{(4)}R_{efgd} = D_b\,K_{ac} - D_a\,K_{bc} = 2D_{[b}\,K_{a]c}.$$

- Alternating [and using $a_b = D_b \ln\alpha$] - the Ricci equation:

$$\gamma^e_a n^b \gamma^f_c n^d\,{}^{(4)}R_{ebfd} = \mathcal{L}_n\,K_{ac} + \frac{1}{\alpha}D_a D_c\alpha + \,K_{ad}\,K^d_c.$$

We will derive each of these in turn.

Gauss equation: By definition of Riemann, for any vector $U^h$.
$\gamma^e_a\gamma^f_b\gamma^g_c U^h\,{}^{(4)}R_{efgh} = 2\gamma^e_a\gamma^f_b\gamma^g_c\nabla_{[e}\nabla_{f]}U_g$ Now take $U^h$ spatial.
Want to use $\gamma^c_a\nabla_c U^b = D_a U^b - n^b K_{ac}U^c$ as derived before.

$$\gamma^e_a\gamma^f_b\gamma^g_c\nabla_{[e}\nabla_{f]}U_g = \underbrace{\gamma^e_{[a}\gamma^f_{b]}\gamma^g_c}_{\text{Just carry this.}}\,\nabla_e\nabla_f U_g = \gamma^e_{[a}\gamma^f_{b]}\gamma^g_c\nabla_e(\gamma^h_f\nabla_h U_g) - \underbrace{\gamma^e_{[a}\gamma^f_{b]}\gamma^g_c[\nabla_e(n_f n^h)]\nabla_h U_g}_{(*)}$$

Shifting antisymmetrization more convenient for following manipulation.

Using compatibility of $g_{ab}$ with $\nabla_a$.

Work on (*):

$-\gamma^e_{[a}\gamma^f_{b]}\gamma^g_c[\nabla_e(n_f n^h)]\nabla_h U_g = -\gamma^e_{[a}\gamma^f_{b]}\gamma^g_c[(\nabla_e n_f)n^h + (\nabla_e n^h)n_f]\nabla_h U_g$ [2nd term 0 due to $n \perp \gamma$]
$= -\gamma^e_{[a}\gamma^f_{b]}\gamma^g_c\nabla_e n_f\; n^h\nabla_h U_g = \gamma^g_c K_{[ab]}n^h\nabla_h U_g = 0$ Back to Gauss:

$$\begin{aligned}\gamma^e_a\gamma^f_b\gamma^g_c U^h\,{}^{(4)}R_{efgh} &= 2\gamma^e_{[a}\gamma^f_{b]}\gamma_{cg}\nabla_e(\gamma^h_f\nabla_h U^g) = 2\gamma^e_{[a}\gamma^f_{b]}\gamma_{cg}\nabla_e(D_f U^g - n^g K_{fh}U^h)\\ &= 2D_{[a}D_{b]}U_c - 2\gamma^e_{[a}\gamma^f_{b]}\gamma_{cg}(\nabla_e n^g)K_{fh}U^h = R_{abcd}U^d + 2K_{c[a}K_{b]h}U^h\end{aligned}$$

As we wanted, since $U^a$ is an arbitrary spatial tensor.

Codazzi-Mainardi equation: Use definition of Riemann as before.
$\gamma^e_a\gamma^f_b\gamma^g_c n^h\,{}^{(4)}R_{efgh} = 2\gamma^e_a\gamma^f_b\gamma^g_c\nabla_{[e}\nabla_{f]}n_g = -2\gamma^e_a\gamma^f_b\gamma^g_c\nabla_{[e|}(K_{|f]g} + n_{|f]}a_g) = -2D_{[a}K_{b]c} + 2\gamma^g_c a_g K_{[ab]}$
[Last term is zero]. ☐

Ricci equation: Lemma: Acceleration of Eulerian observers [normal, propagate perpendicular to the hypersurfaces]

$$\begin{aligned}a_b &= \gamma^c_b n^d\nabla_d n_c = \alpha\gamma^c_b\nabla_d(\alpha^{-1}\,n_c)n^d = \alpha^2\gamma^c_b(\nabla_d\nabla_c t)\nabla^d t = \alpha^{-1}\gamma^c_b\nabla_c[\{-\nabla_a t\nabla^a t\}^{-1/2}] = \alpha^{-1}D_b\alpha\\ &= D_b\ln\alpha\end{aligned}$$

[Used $\gamma^b_a n_b = 0$, $a_b$ spatial since $n^b\nabla_a n_b = 0$ since $n^a n_a = -1$.]
→ $\nabla_a n_b = -K_{ab} - n_a a_b = -K_{ab} - n_a D_b\ln\alpha$ .

Ok, now some pain:

$$\begin{aligned}\gamma^e_a n^f \gamma^g_b n^h \, {}^{(4)}R_{efgh} &= 2\gamma^e_a n^f \gamma^g_b \nabla_{[e}\nabla_{f]} n_g = -2\gamma^e_a n^f \gamma^g_b \nabla_{[e|}(K_{|f]g} + n_{|f]} D_g \ln\alpha) \\ &= -\gamma^e_a n^f \gamma^g_b \nabla_e K_{fg} + \gamma^e_a n^f \gamma^g_b \nabla_f K_{eg} + \gamma^e_a \gamma^g_b \nabla_e (D_g \ln\alpha) + \gamma^e_a n^f \gamma^g_b (\nabla_f n_e)(D_g \ln\alpha) \\ &= \gamma^e_a K_{fg} \gamma^g_b \nabla_e n^f + \gamma^e_a n^f \gamma^g_b \nabla_f K_{eg} + D_a(D_b \ln\alpha) + D_a(\ln\alpha) D_b(\ln\alpha) \\ &= -K_a{}^f K_{bf} + n^f \nabla_f K_{ab} + 2n_{(a|} n^e n^f \nabla_f K_{|b)e} + n_a n_b n^e n^f n^g \nabla_f K_{eg} \\ &\quad + D_a(\alpha^{-1} D_b \alpha) + \alpha^{-2}(D_a\alpha)(D_b\alpha) \\ &= -K_a{}^c K_{cb} + n^f \nabla_f K_{ab} - \; 2n_{(a|} K_{|b)e} n^f \nabla_f n^e \; - n_a n_b n^f n^g K_{eg} \nabla_f n^e (\to 0) + \alpha^{-1} D_a D_b \alpha\end{aligned}$$

$$n^a K_{ab} = 0 \to n^a \nabla_c K_{ba} = -K_{ba} \nabla_c n^a.$$

But look: $-2n_{(a|} K_{|b)e} n^f \nabla_f n^e = -2n_{(a|} K_{|b)e} a^e = 2K_{e(a|}[K_{|b)}{}^e + \nabla_{|b)} n^e]$

Recall Lie derivative: $\mathcal{L}_n K_{ab} = n^c \nabla_c K_{ab} + 2K_{c(a} \nabla_{b)} n^c$

$$\begin{aligned}\text{So: } \gamma^e_a n^f \gamma^g_b n^h \, {}^{(4)}R_{efgh} &= -K_a{}^c K_{cb} + n^f \nabla_f K_{ab} + 2K_{e(a|}[K_{|b)}{}^e + \nabla_{|b)} n^e] + \alpha^{-1} D_a D_b \alpha \\ &= \mathcal{L}_n K_{ab} + \alpha^{-1} D_a D_b \alpha + K_a{}^c K_{cb} \qquad \square\end{aligned}$$

BREAK

So far: 3+1 decomposition of geometry. Now to GR!

Einstein equations: $G_{ab} = 8\pi T_{ab}$ with $G_{ab} = {}^{(4)}R_{ab} - \frac{1}{2} g_{ab} {}^{(4)}R$.

Decompose S-E tensor: $\rho = n^a n^b T_{ab}, \quad j_a = -n^b \gamma^c_a T_{bc}, \quad S_{ab} = \gamma^c_a \gamma^d_b T_{cd}.$

${}^{(4)}R_{ab} = {}^{(4)}R^c{}_{acb}$ will need contractions. Let's work these out.

Contract Gauss equation with spatial metric [twice]:

$$\underset{\displaystyle {}^{(4)}R + 2n^a n^{b} {}^{(4)}R_{ab}}{\underset{||}{\gamma^{ac}\gamma^{bd}[\gamma^e_a \gamma^f_b \gamma^g_c \gamma^h_d \, {}^{(4)}R_{efgh}]}} = R + K^2 - K_{ab}K^{ab} = R + K^2 - K_{ij}K^{ij} \to \text{in adapted coordinates.}$$

Scalar Gauss equation: $\underline{{}^{(4)}R + 2n^a n^{b} {}^{(4)}R_{ab} = R + K^2 - K_{ij}K^{ij}}$

Contract Codazzi-Mainardi equation with the spatial metric [once]:

$$\underset{\displaystyle n^a \gamma^{c}_b {}^{(4)}R_{ac}}{\underset{||}{\gamma^{ac}[\gamma^e_a \gamma^f_b \gamma^g_c n^h \, {}^{(4)}R_{efgh}]}} = D_b K - D_a K^a{}_b$$

Contracted Codazzi equation: $\underline{n^a \gamma^{c}_b {}^{(4)}R_{ac} = D_b K - D_a K^a{}_b}$

Hamiltonian constraint:

$$\begin{array}{c} n^a n^b[{}^{(4)}R_{ab} - \tfrac{1}{2}g_{ab}{}^{(4)}R] = n^a n^b 8\pi T_{ab} = 8\pi\rho \\ \shortparallel \\ \tfrac{1}{2}\ \text{LHS Scalar Gauss} \\ \shortparallel \\ \tfrac{1}{2}(R + K^2 - K_{ab}K^{ab}) \end{array}$$

$$\underline{R + K^2 - K_{ab}K^{ab} = 16\pi\rho} \quad - \quad \text{No time derivatives!}$$

Momentum constraint: $n^b\gamma_a^c[{}^{(4)}R_{bc} - \frac{1}{2}g_{bc}{}^{(4)}R] = n^b\gamma_a^c 8\pi T_{bc} = -8\pi j_a$

$$\begin{array}{c} \shortparallel \\ n^b\gamma_a^{c(4)}R_{bc} \quad \text{LHS Contracted Codazzi} \\ \shortparallel \\ D_aK - D_bK^b{}_a \end{array}$$

$$\underline{D_bK^b{}_a - D_aK = 8\pi j_a} \quad - \quad \text{No time derivatives!}$$

Evolution eq. for $\gamma_{ab}$: $\mathcal{L}_n\gamma_{ab} = -2K_{ab}$ Definition of $K_{ab}$. $\left(n^a = \frac{1}{\alpha}\left(t^a - \beta^a\right),\ t^a = \alpha n^a + \beta^a\right)$

$$\begin{aligned} \mathcal{L}_t\gamma_{ab} &= t^c\nabla_c\gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}t^c \qquad \rightarrow \text{This is } \partial_t\gamma_{ij} \text{ in adapted coordinates. Why?} \\ &= (\alpha n^c + \beta^c)\nabla_c\gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}[\alpha n^c + \beta^c] \\ &= \alpha[n^c\nabla_c\gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}n^c] + [\beta^c\nabla_c\gamma_{ab} + 2\gamma_{c(a}\nabla_{b)}\beta^c] + 2n^c\gamma_{c(a}\nabla_{b)}\alpha(\rightarrow 0) \\ &= \alpha\mathcal{L}_n\gamma_{ab} + \mathcal{L}_\beta\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta\gamma_{ab} \end{aligned}$$

Notice that this would work replacing $\gamma_{ab}$ with any symmetric spatial tensor.

$\underline{\mathcal{L}_t\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta\gamma_{ab}}$ But this is really just rewriting definition of $K_{ab}$. Still need 6 EEs!

Evolution equation for $K_{ab}$: Write EE's in trace-reversed form as: ${}^{(4)}R_{ab} = 8\pi\left(T_{ab} - \frac{1}{2}g_{ab}T\right)$

$$\text{Project: } \gamma_a^c\gamma_b^{d(4)}R_{cd} = \gamma_a^c\gamma_b^d g^{ef(4)}R_{cedf} = \underbrace{\gamma_a^c\gamma_b^d\gamma^{ef(4)}R_{cedf}}_{\text{Use Gauss equation.}} - \underbrace{\gamma_a^c\gamma_b^d n^e n^{f(4)}R_{cedf}}_{\text{Use Ricci equation.}}$$

Putting this together: $\mathcal{L}_nK_{ab} = -\alpha^{-1}D_aD_b\alpha + R_{ab} + KK_{ab} - 2K_a{}^cK_{bc} - 8\pi[S_{ab} - \frac{1}{2}\gamma_{ab}(S-\rho)]$
Same trick on Lie derivative as in previous case:

$$\underline{\mathcal{L}_tK_{ab} = -D_aD_b\alpha + \alpha[R_{ab} + KK_{ab} - 2K_a{}^cK_{bc}] - 8\pi\alpha[S_{ab} - \tfrac{1}{2}\gamma_{ab}(S-\rho)] + \mathcal{L}_\beta K_{ab}}$$

Finally: ADM [better "York"] equations    in adapted coordinates $a \rightarrow i$, $\mathcal{L}_t \rightarrow \partial_t$

$$H = R + K^2 - K_{ab}K^{ab} - 16\pi\rho = 0$$

$$M_a = D_bK^b{}_a - D_aK - 8\pi j_a = 0$$

Prototype "free-evolution" formulation

$$\mathcal{L}_t\gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta\gamma_{ab}$$

In original ADM:

$$\mathcal{L}_tK_{ab} = -D_aD_b\alpha + \alpha[R_{ab} + KK_{ab} - 2K_a{}^cK_{bc}] - 8\pi\alpha[S_{ab} - \tfrac{1}{2}\gamma_{ab}(S-\rho)] + \mathcal{L}_\beta K_{ab} - \tfrac{\alpha}{4}\gamma_{ab}H$$

Counting:

4-metric: 10 - 4 constraints - 4 "gauge" d.o.f = 2 → dynamical degrees of freedom
3-metric + extrinsic curvature: 12 - 4 constraints - 4 "gauge" d.o.f = 4 →
2 for spatial metric and 2 for extrinsic curvature.

Comparison with the Maxwell equations: writing them in flat space, with vector potential:

$$\begin{aligned} D_i E^i &= 4\pi\rho_E && \rightarrow \text{constraint} \\ \partial_t A_i &= -E_i - \partial_i \Phi && \rightarrow \text{evolution equations} \\ \partial_t E_i &= -\Delta A_i + D_i D^j A_j - 4\pi j_K && \end{aligned}$$

Analogy: $A_i \rightarrow \gamma_{ij}$, $\Phi \rightarrow \beta^i$, $E_i \rightarrow K_{ij}$

Hamiltonian formulation: [See Alcubierre, p. 75, 80-81] Since we're always talking about the ADM equations [Arnowitt, Deser, Misner], let's at least say what ADM really did.
Lagrangian density for GR: $\mathcal{L} = \sqrt{-g}\,{}^{(4)}R = \alpha\sqrt{\gamma}\,(R + K_{ab}K^{ab} - K^2)$, with $\gamma$ det. of 3-metric.
Canonical momentum conjugate to $\gamma_{ab}$: $\pi^{ab} = \frac{\delta\mathcal{L}}{\delta\dot{\gamma}_{ab}} = \sqrt{\gamma}\,[K\gamma^{ab} - K^{ab}]$, with $\dot{\gamma}_{ab} = \mathcal{L}_t\gamma_{ab}$.
Lapse and shift have no canonical momenta attached, so they're not dynamical variables.
Hamiltonian density in normal way by Legendre transform: $\mathcal{H} = \pi^{ab}\dot{\gamma}_{ab} - \mathcal{L}$. Total Hamiltonian:

$$H = -\int_{\Sigma_t} (\alpha \underbrace{C_H}_{\text{Hamiltonian constraint}} - 2\beta^a \underbrace{C^H_a}_{\text{Momentum constraint}})\sqrt{\gamma}\,d^3x \qquad \text{"vanishes on shell"}; \qquad K_{ab} = -\frac{1}{\sqrt{\gamma}}\left(\pi_{ab} - \frac{1}{2}\gamma_{ab}\pi\right)$$

Equations of motion (by variation): $\dot{\gamma}_{ab} = \frac{\delta H}{\delta\pi^{ab}}$, $\dot{\pi}^{ab} = -\frac{\delta H}{\delta\gamma_{ab}}$. Equivalently by Poisson bracket.
ADM formulation [?] (derived from the Einstein tensor) differs in a simple way from the "York" formulation [?] (derived from the Ricci tensor).

## 2 PDEs: well-posedness and hyperbolicity

PDEs of physics: The physicist's intuition:
In classical mechanics the motion of physical quantities is universally described by PDEs. These PDEs can be characterized most simply as either elliptic, parabolic or hyperbolic.
First simplest version: Consider 2nd order, linear PDE with constant coefficients:
$a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} +$ "lower order terms" $= 0$, with $a^2 + b^2 + c^2 > 0$.

- Elliptic: $b^2 - ac < 0$
  No intrinsic "time", good BVP-model, ex: Laplace eqn.
  Constraints: [Standard form]. $H = R + K^2 - K_{ij}K^{ij}$, $M_i = D^j(K_{ij} - \gamma_{ij}K)$
- Parabolic: $b^2 = ac$
  Intrinsic time, good IVP, infinite propagation speeds, ex: heat equation.
  (Apparent horizons). (Some methods).
- Hyperbolic: $b^2 - ac > 0$
  Time - and causality: i.e. finite propagation speeds. Fundamental particularly in relativistic context. Ex: wave eqn. ADM evolution equations [in some sense]
  $\partial_t\gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta\gamma_{ij}$, $\partial_t K_{ij} = -D_iD_j\alpha + \alpha[R_{ij} + KK_{ij} - 2K_i{}^kK_{jk}]\mathcal{L}_\beta K_{ij}$

Life is complicated! Models in nature arise with all types - especially in theories with "gauge freedom" like E&M, GR. Encounter all three types!
Type of problem (IVP, IBVP, BVP) determined by classification.

Well-posedness: A PDE problem is called well-posed if there exists a unique solution that depends continuously (on some norm) on given data.
"Change initial data a little, outcome changes a little".

Hyperbolic PDE systems in first order form [Kreiss Busenhart]
Consider a system of PDEs of the form $\partial_t U = A^p\partial_p U + S$ (1), with $U(t, x^i) = U \in \mathbb{R}^n$ and where $A^p$ in as $(n \times n)$ matrix $\forall p$. [constant]
Cauchy/IVP: Specify $U(t = 0, x^i)$. What is the solution $U(t, x^i)$?
PDE problem well-posed if there's a norm $||\cdot||$ such that $||U(t,\cdot)|| \le Ke^{\alpha t}||U(0,\cdot)||$ with $K$ and $\alpha$ constants independent of initial data. ($||\cdot||$, (i) $||aU|| = |a|\,||U||$, (ii) triangle inequality ($||U + V|| \le ||U|| + ||V||$), (iii) $||U|| = 0 \Leftrightarrow U = 0$).

Example of ill-posed IVP: 2D Laplace equation: $\partial_t^2\phi = -\partial_x^2\phi$.
First order reduction: $\partial_t\phi = U_1$, $\partial_x\phi = U_2 \to \partial_t U_1 = -\partial_x U_2$, $\partial_t U_2 = \partial_x U_1$.
Choose ID: $\phi(0, x) = e^{ikx}\phi_0 \to$ Solution: $\phi = \phi_0 e^{kt+ikx}$ [Cheating! But you can take the real part.] $U_1 = k\phi_0 e^{kt+ikx}$, $U_2 = ik\phi_0 e^{kt+ikx}$ Exponential growth dependent on initial data. [Code exercise].

Example 2: Weakly hyperbolic model problem: $\partial_t\begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\partial_x\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$, with $U =$

$(U_1, U_2)^T$
Initial data: $U(0,x) = (Be^{ikx}, Ae^{ikx})^T \quad \Rightarrow U_1(t,x) = (ikAt + B)e^{ik(t+x)}$, $U_2(t,x) = Ae^{ik(t+x)}$.
$U_2$ is fine (oscillates in time → bounded), $U_1$ presents a linear growth but rate depends on ID.
IVP ill-posed. [Code exercise].

So what does work?

### Weak, strong and symmetric hyperbolicity

Consider IVP for (1). Take arbitrary unit vector $s^i$. The principal symbol in the $s^i$ direction is $A^s = A^i s_i$.

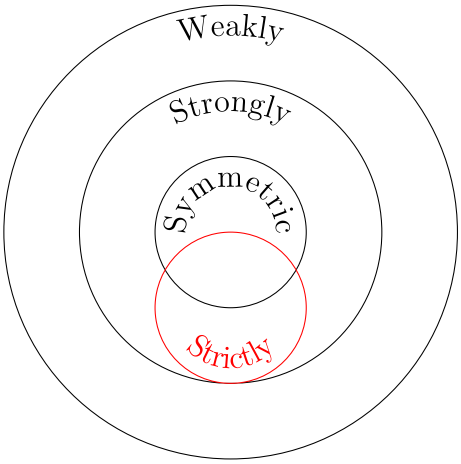
Defn: If $\forall s^i$ the principal symbol has real eigenvalues, the system is called weakly hyperbolic.

Defn: If $\forall s^i$ the principal symbol has real eigenvalues and a complete set of eigenvectors, and $|T_s| + |T_s^{-1}| \leq K$ (with $K$ independent of $s^i$ and $T_s$ has eigenvectors of $p^s$ as columns) holds, then the system is strongly hyperbolic.

Defn: If there exists a symmetric (Hermitian, or self-adjoint, complex square matrix equal to its own complex-conjugate transpose: for $A$ Hermitian, $a_{ij} = \overline{a_{ji}}$ or $A = \overline{A^T} = A^H = A^+$), positive definite matrix $H$ (independent of $s_i$), called a symmetrizer, such that $HA^p$ is symmetric (Hermitian) $\forall p$, then the system is called symmetric hyperbolic.

Diagram: (for systems of the form of (1))

Strict hyperbolicity: all eigenvalues real and distinct.



Intuitive summary:

- Symmetric hyperbolicity: good IBVP (depending on bcs).
- Strong hyperbolicity: good IVP, IBVP harder.
- Weak hyperbolicity: nothing!

Theorem: IVP for (1) is wellposed iff the system is strongly hyperbolic.
(Part of) Proof: Apply Fourier transform $\hat{f} = \int_{-\infty}^{\infty} f e^{2\pi i \omega x} dx$ to (1): $\partial_t \hat{U} = i|\omega| A^{\omega'} \hat{U}$

Strong hyperbolicity ⇔ ∃ similarity transformation such that $S(\omega')A^{\omega'}S^{-1}(\omega') = \Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$

$S^{-1}(\omega')$ is matrix of eigenvectors as columns $T_s$ above.
Let $\hat{H}(\omega') = S^+(\omega')S(\omega')$. $\hat{H}$ is symmetric and positive definite. (Sylvester's law of inertia).
Then $\hat{H}A^{\omega'} = (\hat{H}A^{\omega'})^+$
Show it: $\hat{H}A^{\omega'} - (\hat{H}A^{\omega'})^+ = S^+SA^{\omega'} - (A^{\omega'})^+S^+S = S^+(S(A^{\omega'})S^{-1} - (S^+)^{-1}(A^{\omega'})^+S^+)S =$

$S^{+}(\Lambda - \Lambda^{+})S = 0$

Consider the norm in Fourier space: $||\hat{U}||^2_{\hat{H}} = \int \hat{U}^{+}\hat{H}\hat{U}d\omega$

Parseval-Plancherel identity ($\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\, d\omega$) & $|T_s|$ condition guarantees that this is a norm in physical space equivalent to $L_2$. Compute time der:

$$\partial_t ||\hat{U}||^2_{\hat{H}} = \int_{-\infty}^{\infty} [\hat{U}^{+}\hat{H}A^{\omega'}i\hat{U}|\omega| - i(A^{\omega'}\hat{U})^{+}\hat{H}\hat{U}|\omega|]d\omega = \int_{-\infty}^{\infty} i|\omega|\hat{U}^{+}[\hat{H}(\omega')A^{\omega'} - (\hat{H}(\omega')A^{\omega'})^{+}]\hat{U}d\omega = 0$$

Norm is conserved! ⇒ System is well-posed! Note: source terms do not break the estimate. Worst growth possible is exponential.

This is for linear, constant coefficient systems. Long way from GR. But: linearizing about an arbitrary solution, these results carry over for local in time well-posedness. [Also FT2S!]

BREAK

Consider example: The ADM equations with fixed unit lapse and zero shift.

Equations of motion are: $\partial_t \gamma_{ij} = -2K_{ij}, \quad \partial_t K_{ij} = R_{ij} + KK_{ij} - 2K_i{}^k K_{jk}$

First order reduction: $\partial_k \gamma_{ij} = \Phi_{kij}$, so $R_{ij} \simeq -\frac{1}{2}\partial_k \Phi^k{}_{ij} + 2\partial_{(i}\Phi^k{}_{j)k} - \frac{1}{2}\partial_{(i}\Phi_{j)k}{}^k$,

where $\simeq$ means "up to lower order derivatives".

So: $\partial_t \gamma_{ij} \simeq 0, \quad \partial_t \Phi_{kij} \simeq \partial_k K_{ij}, \quad \partial_t K_{ij} \simeq -\frac{1}{2}\partial_k \Phi^k{}_{ij} + 2\partial_{(i}\Phi^k{}_{j)k} - \frac{1}{2}\partial_{(i}\Phi_{j)k}{}^k$

This is a first order PDE system. We need to know the principal symbol, but the indices are a pain. Therefore we make a convenient choice of variables, so that in the end we have to deal with small matrices.

- Define: $\perp^i_j = \delta^i_j - s^i s_j$ ($s^i$ is unit spatial vector)
- Linearize equations around flat space. (Write $\eta_{ij}$ for "metric" and use $\gamma_{ij}$, $\Phi_{kij}$, $K_{ij}$ for the perturbations).
- $\gamma_{ij}$ is decoupled, so we can ignore it.

Choose: ($A$ stands for transverse)

$\Phi_{sss} \equiv s^i s^j s^k \Phi_{ijk}, \quad \Phi_{sqq} \equiv s^i \perp^{jk} \Phi_{ijk}, \quad \Phi_{qqs} \equiv \perp^{ij} s^k \Phi_{ijk}, \quad \Phi_{ssA} \equiv s^i s^j \perp^k_A \Phi_{ijk}, \quad \Phi_{qqA} \equiv \perp^{ij} \perp^k_A \Phi_{ijk}$

$K_{ss} \equiv s^i s^j K_{ij}, \quad K_{qq} \equiv \perp^{ij} K_{ij}, \quad K_{sA} \equiv s^i \perp^j_A K_{ij}, \qquad \Phi^{TF}_{sAB} \equiv (\perp\perp - \frac{1}{2}\perp\perp)\Phi_{ijk}, \quad K^{TF}_{AB} \equiv (\ )K_{ij},$

$\tilde{\Phi}_{Aij} \equiv$ remaining components of $\Phi$

Now if we 2+1 split derivatives in $s^i$: $\partial_i U = s_i s^j \partial_j U + \perp^j_i \partial_j U$ , then we can write:

Exercise: fill in as many of the missing steps you need to be convinced.

$$\begin{cases} \tilde{U}_A = (\tilde{\Phi}_{Aij}) \\ U^{TF}_{AB} = (\Phi^{TF}_{sAB}, K^{TF}_{AB})^{+} \\ U_A = (\Phi_{qqA}, \Phi_{ssA}, K_{sA})^{+} \\ U_s = (\Phi_{qqs}, \Phi_{sss}, \Phi_{sqq}, K_{ss}, K_{qq})^{+} \end{cases} \quad \rightarrow \quad \begin{cases} \partial_t \tilde{U}_A \simeq 0 \\ \partial_t U^{TF}_{AB} = A^s_{(1)}\partial_s U^{TF}_{AB} + \text{“}\partial_A \text{ derivatives''} \\ \partial_t U_A = A^s_{(2)}\partial_s U_A + \text{“}\partial_A \text{ derivatives''} \\ \partial_t U_s = A^s_{(3)}\partial_s U_s + \text{“}\partial_A \text{ derivatives''} \end{cases}$$

where

$$A^s_{(1)} = \begin{pmatrix} 0 & -2 \\ -\frac{1}{2} & 0 \end{pmatrix} \qquad \text{tensor block (GWs)} \qquad \underline{\pm 1, \text{ complete set}}$$

$$A^s_{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \qquad \text{vector block} \qquad \underline{0, \pm 1, \text{ missing one eigenvector!}}$$

$$A^s_{(3)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 \end{pmatrix} \qquad \text{scalar block} \qquad \underline{0, \pm 1, 0, 0, \text{ missing two eigenvectors!}}$$

Eigenvalues and eigenvectors? 5 minutes with Mathematica: ↑
Conclusion: ADM is only weakly hyperbolic (with this gauge and reduction)!

Why is this calculation unsatisfactory? (Besides the sad conclusion!)

1. Linearization? Is this ok?
2. Reduction? $\partial_k \Phi_{ijl} \overset{?}{=} \partial_i \Phi_{kjl} \to$ constraint!
3. Gauge choice? Might some other gauge be ok?

Still, this was ignored for $\sim 30$ years. Is there a strongly hyperbolic formulation of GR?

Two main free evolution formulations in NR:

### The Generalized Harmonic Gauge (GHG) Formulation

Consider the 4D Ricci tensor with $\Gamma_a = g^{bc}\Gamma_{abc}$:

$$R_{ab} \simeq -\frac{1}{2} \underbrace{g^{cd}\partial_c \partial_d g_{ab}}_{\text{“Like” wave operator}} + \underbrace{\partial_{(a}\Gamma_{b)}}_{\substack{\text{Contains second derivatives.} \\ \text{How can we get rid of this?}}} + \text{ lower order terms}$$

Note that $\Gamma_a = -g_{ab}\Box x^b$, with $x^b$ local coords. So choose $\Box x^b = 0$ (“harmonic coords”), then

$$R_{ab} \simeq -\frac{1}{2} \underbrace{g^{cd}\partial_c \partial_d g_{ab}}_{\text{This is just a wave operator, like in the wave equation! } \Rightarrow \text{ The resulting system is strongly hyperbolic!}} + \text{ lower order terms}$$

To be more formal: for $\Box x^b = 0$ [$= H^b(g, x)$ more generally], define constraint $Z_a = -\Gamma_a \doteq 0$:

Solve: $R_{ab} + \partial_{(a}Z_{b)} \simeq -\frac{1}{2}g^{cd}\partial_c\partial_d g_{ab} +$ lower order terms

$\partial_t Z_a =$ Hamiltonian and momentum constraint + $\partial_i Z_a$ terms.

If $Z_\mu = 0$ at $t = 0$ and constraints of GR satisfied → solution is solution to GR.

Generalized Harmonic Gauge (GHG) Formulation:

- symmetric hyperbolic
- all speeds are the light speed (up to first order reduction)
- well-posed IBVP
- finite difference or pseuspectral codes [codes: Pretorius, SpEC]
- Gauge: $\Box x^\mu = H^\mu \quad \overset{3+1}{\Rightarrow} \quad \begin{cases} \partial_t\alpha = -\alpha^2(K+F) + \mathcal{L}_\beta\alpha \\ \partial_t\beta^i = \alpha^2\Gamma^i - \alpha\partial^i\alpha + F^i + \beta^j\partial_j\beta^i \end{cases}$ with $F,\ F^i$ free
- black hole excision, basic idea: cut BH region out of numerical domain inside of apparent horizon, Boundary "should be" outflow ⇒ no bcs needed. Then move excision region or carefully move coordinates.

"Moving-puncture" / Conformally Decomposed Formulations:

- BSSN/BSSNOK [0-speed mode], Z4c/CCZ4 [no 0-speed mode, better!]
- well-posed IVP since it's strongly hyperbolic
- progress on well-posedness of IBVP, hard to implement.
- radiation controlling constraint-preserving boundary conditions implemented.
- finite differences (almost universally) [codes: Einstein Toolkit, BAM, Lean, NRPy+]
- "Moving-puncture" gauge: $\begin{cases} \partial_t\alpha = -2\alpha^2 K + \mathcal{L}_\beta\alpha \\ \partial_t\beta^i = \mu_s\Gamma^i - \eta\beta^i + \beta^j\partial_j\beta^i \end{cases}$ (can use for BNS spacetimes!)
- "Moving-puncture method" [wormhole vs. trumpet ID diagram], basic idea: singular part of the back hole geometry is encoded in a spatial conformal factor. There's a bad point at the puncture, but clever choice of evolved varables makes point "manageable" numerically. Puncture then advected around by the moving-puncture conditions.

# 3 Initial data - solving the constraints [Alcubierre]

In the 3+1 decomposition we arrived at evolution equations and constraints. We now have an idea of how to make evolution equations “nice”. Need initial data! (12 DoF)

$$\left.\begin{aligned} H &= R + K^2 - K_{ij}K^{ij} \\ M_i &= D^j(K_{ij} - \gamma_{ij}K) \end{aligned}\right\} \rightarrow \text{4 eqs., but haven't stated } \underline{\text{what}} \text{ to solve for, “not even posed”.}$$

Issues? (i). Get good PDEs. (ii). Choose data to model physics we’re interested in. Earliest approaches focused mostly in (i) and so will we.

<u>York-Lichnerowicz conformal decomposition</u>: What do we solve for?
$\gamma_{ij} = \psi^4\tilde{\gamma}_{ij}$, with $\psi$ the conformal factor and $\tilde{\gamma}_{ij}$ the conformal background metric - natural to choose $\tilde{\gamma} = 1$ (i.e. $\psi^4 = \gamma^{1/3}$), but not needed.
Plug into the Hamiltonian constraint:

$$\tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{8}\psi^5K^2 + \frac{1}{8}\psi^5K_{ij}K^{ij} = -2\pi\psi^5\rho \tag{2}$$

with $\tilde{D}_i$ the covariant derivative and $\tilde{R}$ the Ricci scalar associated with $\tilde{\gamma}_{ij}$. (2) is quasilinear elliptic equation for $\psi$. Spatial metric $\gamma_{ij}$ naturally constrained.

Now onto the momentum constraint: $K_{ij} = \underbrace{A_{ij}}_{\text{Tracefree}} + \frac{1}{3}\gamma_{ij}\underbrace{K}_{\text{Trace}}$

Covariantly (in spatial slice) decompose $S^{ij} = (LX)^{ij} + T^{ij}$, where $S^{ij}$ is symmetric tracefree, $T^{ij}$ is symmetric, tranverse-traceless ($D_iT^{ij} = 0$, $T^i_i = 0$), and $(LX)^{ij} = D^iX^j + D^jX^i - \frac{2}{3}\gamma^{ij}D_kX^k$.
$S^{ij}$: arbitr. sym. TF, $T^{ij}$: TT part of $S^{ij}$, $(LX)^{ij}$: longitudinal part of $S^{ij}$, *conf. Killing form.*

How to obtain an elliptic equation from this decomposition and momentum constraint? Two options to construct $L$ from, either use:

- Conformal metric (and associated covariant derivative).
- Physical metric (likewise).

<u>Conformal transverse traceless decomposition</u>
Define $A^{ij} = \psi^{-10}\tilde{A}^{ij}$, $[A_{ij} = \psi^{-2}\tilde{A}_{ij}]$. Take care about metric used!
$\tilde{A}^{ij} = (\tilde{L}X)^{ij} + \tilde{Q}^{ij}$, with $(\tilde{L}X)^{ij}$, $\tilde{Q}^{ij}$ defined with respect to $\tilde{D}_i$.

Momentum constraint: $D^j(K^i_j - \gamma^i_jK) = 8\pi j^i$
Substitute the previous definitions: $\tilde{\Delta}_LX^i = \frac{2}{3}\psi^6\tilde{D}^iK + 8\pi\psi^{10}j^i$, with

$$\tilde{\Delta}_LX^i \equiv \tilde{D}_j(\tilde{L}X)^{ij} = \tilde{D}^j\tilde{D}_jX^i + \frac{1}{3}\tilde{D}^i(\tilde{D}_jX^j) + \tilde{R}^{ij}X_j \tag{3}$$

where you use $D_jS^{ij} = \psi^{-10}\tilde{D}_j(\psi^{10}S^{ij})$ and $\tilde{Q}^{ij}$ is transverse ($\tilde{D}_j\tilde{Q}^{ij} = 0$).
Want to choose a method so that we choose a symmetric tracefree tensor, since “tranverse” is differential and this is more of a pain. But: for $\tilde{Q}^{ij} = \tilde{M}^{ij} - (\tilde{L}Y)^{ij}$, $\tilde{M}^{ij}$ is sym. TF tensor.

$\tilde{L}$ operator is linear, so: $\tilde{A}^{ij} = (\tilde{L}V)^{ij} + \tilde{M}^{ij}$ with $V^i = X^i - Y^i$. $\tilde{\Delta}_L$ also linear, so momentum constraint with (3) becomes: $\tilde{\Delta}_L V^i = \frac{2}{3}\psi^6 \tilde{D}^i K - \tilde{D}_j \tilde{M}^{ij} + 8\pi\psi^{10} j^i$

Summary: $\quad \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij}, \qquad K^{ij} = \psi^{-10}\tilde{A}^{ij} + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K, \qquad \tilde{A}^{ij} = (\tilde{L}V)^{ij} + \tilde{M}^{ij}$

$$\tilde{\Delta}_L V^i - \frac{2}{3}\psi^6 \tilde{D}^i K = -\tilde{D}_j \tilde{M}^{ij} + 8\pi\psi^{10} j^i, \qquad \tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

BREAK

### Physical transverse traceless decomposition

$A^{ij} = (LW)^{ij} + Q^{ij}$, where $Q^{ij}$ is transverse-traceless with respect to $\gamma_{ij}$.
Momentum constraint: $\tilde{\Delta}_L W^i + 6(\tilde{L}W)^{ij}\tilde{D}_j(\ln\psi) = \frac{2}{3}\tilde{D}^i K + 8\pi\psi^4 j^i$, where $(LW)^{ij} = \psi^{-4}(\tilde{L}W)^{ij}$.

Again, it is annoying if the "free data" $Q^{ij}$ has to satisfy a differential constraint.
So: $Q^{ij} = \tilde{M}^{ij}\psi^{-10} - (LZ)^{ij}$, $Q$ is transverse. $\Rightarrow \tilde{\Delta}_L Z^i = 6(\tilde{L}Z)^{ij}\tilde{D}_j(\ln\psi) = \psi^{-6}\tilde{D}_j\tilde{M}^{ij}$.

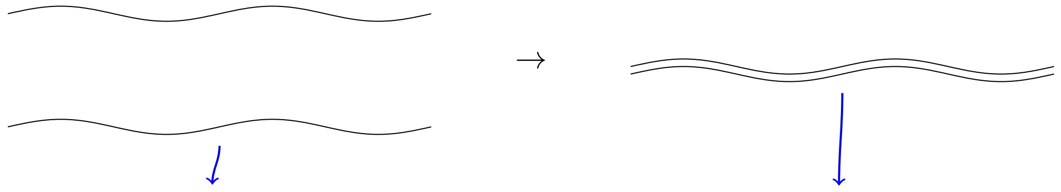
Again: $V^i = W^i - Z^i$. Total decomposition (summary):
$\gamma_{ij} = \psi^4\tilde{\gamma}_{ij}, \qquad K^{ij} = \psi^{-4}\left(\tilde{A}^{ij} + \frac{1}{3}\tilde{\gamma}^{ij}K\right), \qquad \tilde{A}^{ij} = (\tilde{L}V)^{ij} + \psi^{-6}\tilde{M}^{ij}$

$$\tilde{\Delta}_L V^i + 6(\tilde{L}V)^{ij}\tilde{D}_j(\ln\psi) = \frac{2}{3}\tilde{D}^i K - \psi^{-6}\tilde{D}_j\tilde{M}^{ij} + 8\pi\psi^4 j^i, \; \tilde{\Delta}\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5 K^2 + \frac{1}{8}\psi^5\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

Both conformal and physical TT-decompositions give a method for the constraints - but how are we supposed to choose data to represent a particular physical scenario? We could:

- Simplify the form of the constraints with careful assumptions.
- Expand system to solve with "easier" given data. Start here.

### Conformal thin sandwich (CTS) equations [York'99]



"Thin sandwich", old approach of '60's, Misner (et al.).

Choose $\dot{u}$ for some variables, instantaneous control of dynamics.

$\tilde{u}_{ij} = \partial_t\tilde{\gamma}_{ij}$, with $\tilde{\gamma}_{ij}$ is conformal metric like before. Choose

$$\tilde{\gamma}^{ij}\tilde{u}_{ij} = 0 \tag{4}$$

[$\dot{\tilde{\gamma}} = 0$ at $t = 0$, volume element of the metric is momentarily fixed].
Now write

$$\begin{aligned} u_{ij} &= \partial_t\gamma_{ij} - \frac{1}{3}\gamma_{ij}(\gamma^{kl}\partial_t\gamma_{kl}) \qquad (5) \\ &= -2\alpha A_{ij} + (L\beta)_{ij} \qquad \text{with same } L \text{ we had before.} \end{aligned}$$

Exercise: (i): (4)$\Rightarrow \partial_t \ln\psi = \partial_t(\ln\gamma^{1/12})$, (ii) $\Rightarrow \tilde{u}_{ij} = \psi^{-4}u_{ij}$

Now work from (5): $A^{ij} = \frac{1}{2\alpha}\left[\,(L\beta)^{ij} \;-\; u^{ij}\,\right] = \frac{\psi^{-4}}{2\alpha}\left[(\tilde{L}\beta)^{ij} - \tilde{u}^{ij}\right]$,

Same conformal transformation

$\tilde{A}^{ij} = \frac{1}{2\tilde{\alpha}}\left[(\tilde{L}\beta)^{ij} - \tilde{u}^{ij}\right]$ and conformal lapse $\tilde{\alpha} = \psi^{-6}\alpha$ with $\tilde{A}^{ij} = \psi^{10}A^{ij}$ like before.

Hamiltonian constraint: $8\tilde{\Delta}\psi - \psi\tilde{R} + \psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} - \frac{2}{3}\psi^5K^2 + 16\pi\psi^5\rho = 0$ [as before].

Momentum constraint: $\tilde{D}_j\left[\frac{1}{2\tilde{\alpha}}(\tilde{L}\beta)^{ij}\right] - \tilde{D}_j\left[\frac{1}{2\tilde{\alpha}}\tilde{u}^{ij}\right] - \frac{2}{3}\psi^6\tilde{D}^iK - 8\pi\psi^{10}j^i = 0.$

Construction of initial data: solve for $\psi$, $\beta^i$, then: $\gamma_{ij} = \psi^4\tilde{\gamma}_{ij}$, $K^{ij} = \psi^{-10}\tilde{A}^{ij} + \frac{1}{3}\gamma^{ij}K$, $\tilde{A}^{ij} = \frac{1}{2\tilde{\alpha}}\left[(\tilde{L}\beta)^{ij} - \tilde{u}^{ij}\right]$. [Everything else is given].

Extended conformal thin sandwich:

In the last approach we had $\tilde{u}_{ij} \sim \dot{\tilde{\gamma}}_{ij}$ as given data, but introduced $\tilde{\alpha}$. $\tilde{u}_{ij}$ we like, as it has an obvious physical interpretation. $\tilde{\alpha}$? Less clear perhaps. But we could note

$\partial_tK = \beta^i\partial_iK - \Delta\alpha + \alpha\left[A_{ij}A^{ij} + \frac{1}{3}K^2\right] + 4\pi\alpha(S+\rho)$

with $\Delta\alpha = \psi^{-4}[\tilde{\Delta}\alpha + 2\tilde{\gamma}^{ij}\partial_i\alpha\partial_j(\ln\psi)]$ then [algebra] $\Rightarrow$

$\tilde{\Delta}\tilde{\alpha}+\tilde{\alpha}\left[\frac{3}{4}\tilde{R} - \frac{7}{4}\psi^{-8}\tilde{A}^{ij}\tilde{A}_{ij} + \frac{1}{6}\psi^6K^2 + 42\tilde{D}_i(\ln\psi)\tilde{D}^i(\ln\psi)\right]+14\tilde{D}_i\tilde{\alpha}\tilde{D}^i\ln\psi+\psi^{-2}(\partial_tK-\beta^i\partial_iK)-4\pi\tilde{\alpha}\psi^4(S+4\rho) = 0$

Now can choose $\partial_tK$ and $K$ [in case you have better intuition for them].

We want to bash BHs together! How? → Simplify equations to put them in tractable form.

BREAK

Recap CTT: $\gamma_{ij} = \psi^4\tilde{\gamma}_{ij}$, $K^{ij} = \psi^{-10}\tilde{A}^{ij} + \frac{1}{3}\psi^{-4}\tilde{\gamma}^{ij}K$, $\tilde{A}^{ij} = (\tilde{L}V)^{ij} + \tilde{M}^{ij}$

$$\tilde{\Delta}_LV^i - \frac{2}{3}\psi^6\tilde{D}^iK = -\tilde{D}_j\tilde{M}^{ij} + 8\pi\psi^{10}j^i, \qquad \tilde{D}^i\tilde{D}_i\psi - \frac{1}{8}\psi\tilde{R} - \frac{1}{12}\psi^5K^2 + \frac{1}{8}\psi^{-7}\tilde{A}_{ij}\tilde{A}^{ij} = -2\pi\psi^5\rho$$

## Multiple black hole initial data

Time symmetric data: Take $K_{ij} = 0$, then $M_i = 0$ (in vacuum).

Hamiltonian constraint becomes: $8\tilde{\Delta}\Psi - \tilde{R}\psi = 0$.

Choose $\tilde{\gamma}_{ij} = \delta_{ij}$ flat, spatial metric is "conformally" flat $\Rightarrow$ $\tilde{\Delta}\psi = D^2_{flat}\psi = 0$ → Laplace eq.

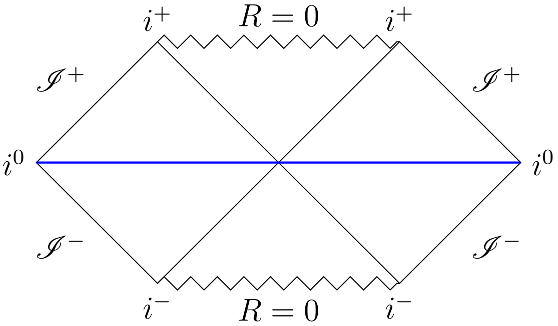
Solution? $\psi = 1 + \frac{M}{2r}$, with $\tilde{ds}^2 = dx^2 + dy^2 + dz^2$ and $r^2 = x^2 + y^2 + z^2$, for $M = 0$ flat space.

In spherical coords: $ds^2 = \left(1 + \frac{M}{2r}\right)^4\left[dr^2 + r^2d\Omega^2\right]$ → spatial metric of Schwarzschild in isotropic coords & standard time slice.

Next solution (Laplace eq is linear): $\psi = 1 + \sum_{i=1}^{N}\frac{M_i}{2|\vec{r}-\vec{r}_i|}$: $N$ black holes intially at rest, $M_i$ "bare masses", $\psi \to \infty$ as $r \to r_i$. Solution known as →

Brill-Lindquist (initial) data:

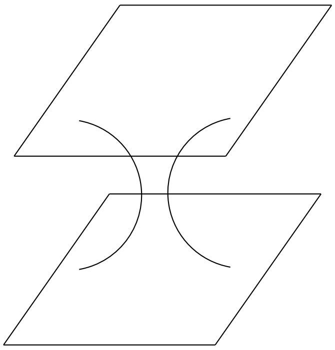
Recall Schwarzschild:

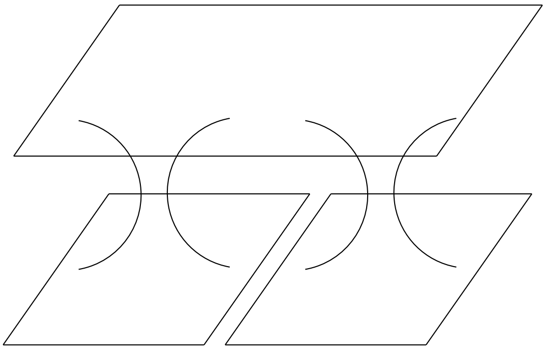


→

Take the blue slice.
↓
Embedding diagram
with two
asymptotically flat
ends.

→





→

$N+1$ asymptotically flat ends - should we worry about the other ends? $r_i$ are not really part of the manifold, known as "punctures" (hidden inside of their horizons).

Brill-Lindquist data:

- May correspond to a BBH is the coordinate separation is large enough.
- Not very relevant for GW astro or astrophysics: BHs are stationary initially (not orbiting).
- Time symmetry is too restrictive! → Give the BHs linear momentum and spin.

Bowen-York extrinsic curvature: Solving the momentum constraint:
Consider the momentum constraint in conformal tranverse traceless decomposition:

$$\tilde{\Delta}_L V^i \quad -\tfrac{2}{3}\psi^6 \tilde{D}^i K \;=\; -\tilde{D}_j \tilde{M}^{ij} \;+\; 8\pi\psi^{10} j^i$$

Choose $\tilde{\gamma}_{ij} = \delta_{ij}$, "conformally flat".

→ 0 for maximal slicing.

→ 0 for free data.

→ 0 in vacuum.

Let's make these solvable. Start with: $\tilde{\Delta}_L \tilde{V}^i = \Delta \tilde{V}^i + \frac{1}{3}\tilde{D}^i \tilde{D}_j \tilde{V}^j = 0$ both terms are flat.
Use Cartesian coords: linear, constant coefficients!
(Bowen&York (again)): $\tilde{V}^i = -\frac{1}{4r}\left[7P^i + n^i(n_j P^j)\right] + \frac{1}{r^2}\epsilon^{ijk} n_j S_k$, where $P^i$, $S^i$ are constant vectors, $\epsilon^{ijk}$ is the Levi-Civita compatible with $\gamma_i$, $n^i$ is the outward pointing unit radial vector.
In vector notation: $\vec{\tilde{V}} = -\frac{1}{4r}\left[7\vec{P} + \vec{n}(\vec{n}\cdot\vec{P})\right] + \frac{1}{r^2}(\vec{n}\times\vec{S})$

Conformal extrinsic curvature:
$\tilde{A}_{ij} = (\tilde{L}\tilde{V})_{ij} = \frac{3}{2r^2}[n_i P_j + n_j P_i + (n_k P^k)(n_i n_j - \delta_{ij})] - \frac{3}{r^3}(\epsilon_{ilk} n_j + \epsilon_{jlk} n_i) n^l S^k$
$K_{ij} = \psi^{-2}\tilde{A}_{ij}$ → Bowen-York extrinsic curvature
Physical iterpretation: $P^i$ → ADM linear momentum [at spatial infinity]; $S^i$ → angular momentum (spin) [at spatial infinity] (not really well defined ...)

"Puncture" initial data:

Now we have an analytic solution for the momentum constraint that represents something like a boosted and/or spinning particle [or several]. What about Hamiltonian constraint? Analytic solution? No such luck ...

$\underbrace{\Delta\psi}_{\text{Flat}} + \frac{1}{8} \underbrace{\psi^{-7}}_{\text{Curse!}} \tilde{A}_{ij}\tilde{A}^{ij} = 0$. As $r \to \infty$ we want $\psi \approx 1 + \frac{M}{2r}$, can take this as bc.

How can we generalize the Brill-Lindquist initial data?

$$\underbrace{\psi}_{\text{Ansatz}} = \underbrace{\psi_{BL}}_{\substack{\psi_{BL} = \sum_{i=1}^{N} \frac{M_i}{2|\vec{r}-\vec{r}_i|} \\ \text{Brill-Lindquist as before}}} + \underbrace{u}_{\text{correction}}$$

Solve for $u$: $\quad \Delta u + \eta \left(1 + \frac{u}{\psi_{BL}}\right)^{-7} = 0$, with $\eta = \frac{1}{8\psi_{BL}^7}\tilde{A}^{ij}\tilde{A}_{ij}$. BC? $u \to 1$ at $\infty$.
What about as $\vec{r} \to \vec{r}_i$? Do we need bc's there? No:

$$\psi_{BL} \sim \frac{1}{|\vec{r}-\vec{r}_i|}, \quad \tilde{A}^{ij}\tilde{A}_{ij} \sim \begin{cases} \frac{1}{|\vec{r}-\vec{r}_i|^6} & \text{Spin} \\ \frac{1}{|\vec{r}-\vec{r}_i|^4} & \text{No spin} \end{cases} \quad \Rightarrow \quad \eta \sim \begin{cases} |\vec{r}-\vec{r}_i| & \text{Spin} \\ |\vec{r}-\vec{r}_i|^3 & \text{No spin} \end{cases}$$

What about $\left(1 + \frac{u}{\psi_{BL}}\right)$? ($u$ finite, but $\psi_{BL}$ blows up). Regular (and $\Delta u \sim 0$ near puncture). But we need to show that there are solutions. Brandt & Brügmann: there are unique $C^2$ solutions $\to$ we can ignore the punctures when solving for $u$.
This is the data that most numerical relativists have been using in applications since 2005. Its accurate numerical solution (with spectral methods) was pioneered by Brügmann, Ansorg, Tichy.

Summary of puncture data:

Strengths:

- Analytic solution for momentum constraint.
- (Partial) Control over physical setup.
- Can be solved really accurately.

Weaknesses:

- No conformally flat slice for Kerr.
- "Junk" radiation: initial data are a spinning BH with radiation.
- Assumptions that are math good are not necessarily physically good.

Further physically relevant improvements:

- Initial data with higher spins: metric cannot be conformally flat → cannot use Bowen-York extrinsic curvature and have to solve 4 coupled elliptic equations.

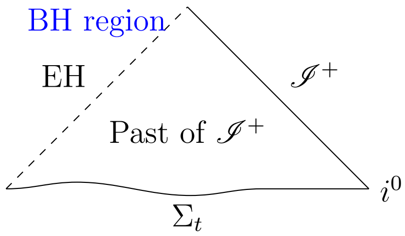
- Eliminate eccentricity: tune initial parameters by running first orbit until eccentricity is small enough.

# 4 Apparent horizons

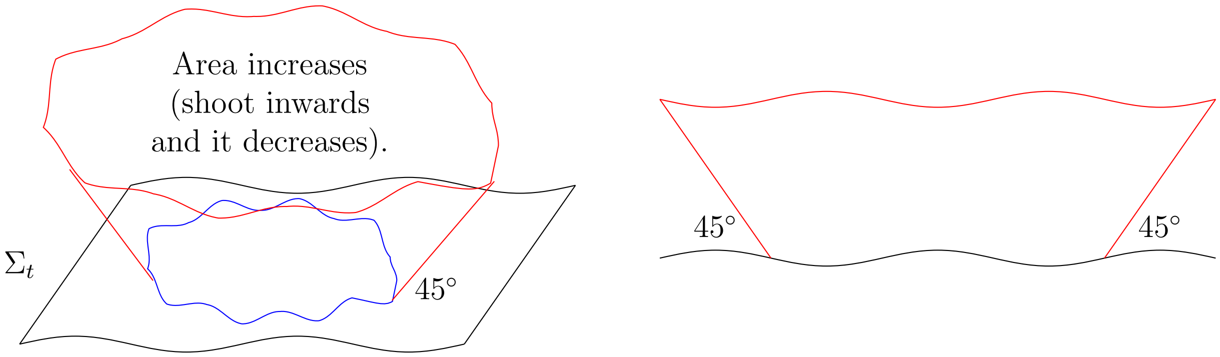
(What we will not talk about:)

Event horizons:
The true definition of a black hole or a black region is the following. Take an asymptoticaly flat spacetime. (Various definitions of this, but future null infinity is a common feature). Consider the complement of the past of future null infinity. This region, if it exists, is called the black hole region. The boundary of the BH region is the event horizon (EH).
We avoid this (when possible) in NR. Why? → The definition is global!
This means that we are required (in principle) to search the whole of spacetime for the EH. Painful. Standard approach: compute a spacetime and look for EH in postprocessing. Requires lots of output. → Reading 4D data. → Interpolation in space and time. Horrible!

1. Evolve until the final BH has settled down.
2. EH is attractor for null geodesics propagating backwards in time.
3. Find EH via backward in time integration of null surfaces (null geodesics enough in spherical symmetry).
4. Null version of the Raychaudhuri equation is suitable to null geodesic congruences.



Enter the apparent horizon (AH):
Think of a sphere at a particular instant of time (intertial) in Minkowski spacetime. Consider what happens to the area of the sphere if we expand it along an outward pointing null vector field. → Use this idea to characterize BH region.

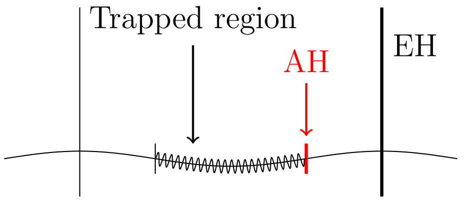


Consider a topological sphere in some other spacetime, again at some instant of time (defined by some time coordinate).
If when we track the area of the sphere along an outward pointing null vector it decreases we

say that the region inside the sphere is trapped.
If the area is constant under this operation, the surface is called "marginaly trapped". The outermost marginally trapped surface (MOTS) (if it exists) is called the apparent horizon.

AH's crucial in proofs of "singularity" theorems.

Clearly this is intuitively consistent with 'nothing escaping', but what is the relationship between the EH & AH?

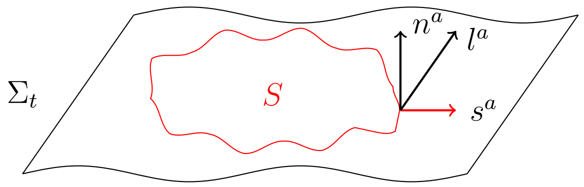


- Assume cosmic censorship. [Big assumption].
- If there is an AH, it must lie inside a spatial slice of the EH.

Is the AH "just as good" as the EH?
No! "Absence of proof is not proof of absence".

- AH depends on spatial slice. Not 4-covariant.
- If you take a weird slice, there may be no AH even in a BH spacetime.
- This can be done even in Schwarzschild. {Wald-Iver}

Mathematical details:
Consider a closed 2D surface $S$ inside a spatial slice $\Sigma_t$. $s^a$ is spatial unit outward normal vector. $n^a$ is timelike unit normal to $\Sigma_t$.
Outgoing null vector: $l^a = n^a + s^a$.
2-metric in $S$: $q^{ab} = g^{ab} + n^a n^b - s^a s^b = \gamma^{ab} - s^a s^b$
Expansion of null-geodesics: $\Theta = +\frac{1}{2} q^{ab} \mathcal{L}_l q_{ab} = +\frac{1}{2} q^{ab} \left( \overset{(1)}{\mathcal{L}_s q_{ab}} + \overset{(2)}{\mathcal{L}_n q_{ab}} \right)$



Meaning? Compare with 3+1 split:
The time derivative of volume form: $\mathcal{L}_n \sqrt{\gamma} = -\sqrt{\gamma} K$ → here area form, but same idea.

($\mathcal{L}_s q_{ab} = 2X_{ab}$, $X$ extrinsic curvature of S as embedded in $\Sigma_t$. $q^{ab} \mathcal{L}_s q_{ab} = 2X$).

But: $\mathcal{L}_s q_{ab} = s^c D_c(\gamma_{ab}(\to 0) - s_a s_b) + 2q_{c(a}D_{b)}s^c \quad \Rightarrow \quad q^{ab}\mathcal{L}_s q_{ab} = 2D_a s^a$ (1)
Now the other term: (2) $q^{ab}\mathcal{L}_n q_{ab} = -2q^{ab}K_{ab} - q^{ab}\mathcal{L}_n(s_a s_b)(\to 0)$

$$\Rightarrow \underline{\Theta = D_i s^i - K + K_{ij}s^i s^j} \tag{6}$$

Equivalent way: $\Theta = q^{ab}\nabla_a l_b = q^{ab}\nabla_a(s_b + n_b) = q^{ab}(D_a s_b - K_{ab}) = D_a s^a - (\gamma^{ab} - s^a s^b)K_{ab}$
Definition of AH? Outermost S with $\Theta = 0$.

"Minimal surface"? Same, with $K_{ij} = 0$. AH can coincide with minimal surface in this case.

How can we characterize / search for the AH?
Level set approach: Suppose AH (in $\Sigma_t$) is a level set of $F(x^i)$. Normal vector: $s^i = \frac{D^i F}{u}$, with $u^2 = \gamma^{ij}(D_i F)(D_j F)$.
(6) $\Rightarrow \Theta = (\gamma^{ij} - u^{-2}(D^i F)(D^j F))(u^{-1}D_i D_j F - K_{ij})$.
Given a slice $\gamma_{ij}$, $K_{ij}$, how can we determine if there is an AH or not?

Examples:

(i) spherical symmetry: $ds^2 = Adr^2 + r^2 B d\Omega^2$, $s^i = (A^{-1/2}, 0, 0)^T$
(6) $\Rightarrow \Theta = \frac{1}{\sqrt{A}}\left(\frac{2}{r} + \partial_r \ln B\right) - 2K^\theta_\theta = 0$. (Algebraic relation).
If this holds, then we have an apparent horizon. E.g: Schwarzschild:
$K^\theta_\theta = 0$, $A = \left(1 - \frac{2M}{r}\right)^{-1}$, $B = 1 \quad \Rightarrow \quad$ AH condition: $\frac{2}{r}\sqrt{\left(1 - \frac{2M}{r}\right)} = 0 \Rightarrow r = 2M$

(ii) axial symmetry: Solve an ODE. Take $F(r, \theta) = r - h(\theta)$, however beware: horizon assumed to be a strahlkörper (ray-body, with rays from the centre intersecting the surface only once).

(iii) "Full" 3D: Various methods, see *Living Review* of Thornburg. Here: Flow method.

Basic idea:

- Introduce an unphysical time $\lambda$.
- Make some guess.
- Then $\partial_\lambda x^i = -\Theta s^i$.
- When $\partial_\lambda x^i = 0$ we have an AH.

Ex: Check in Schwarzschild that the "-" sign is the right way around.

Writing surfaces as $F(x^i, \lambda) = 0$. $\frac{d}{d\lambda}F(x^i, \lambda) = \partial_\lambda F + \frac{dx^i}{d\lambda}D_i F = 0$.
By Flow equation: $\partial_\lambda F = \Theta s^i D_i F \quad \Rightarrow \quad \partial_\lambda F = |DF|\Theta$, since $s^i = \frac{D^i F}{|DF|}$.

Now: Taking $F = r - h(\theta, \varphi) \quad \partial_\lambda h = -|D(r - h)|\Theta$. Parabolic type equation.

- Method is slow.
- Optimization is possible.
- Often "direct" solve still faster.

# 5 Relativistic hydrodynamics

- Most astrophysical systems involve matter sources, which need to be modelled! Fluid approximation: matter is a continuum. "Infinitesimal" fluid element contains many particles.
- Here (follow Alcubierre/most NR groups) Eulerian approach: fix coordinate system (3+1 coordinates). Describe motion in these coordinates.

Special relativistic hydrodynamics

Stress energy tensor (for a perfect fluid: zero viscosity and no heat conduction):
$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\ \eta_{\mu\nu}$, $\qquad p = 0$ "dust".
$u^\mu$: 4-velocity of fluid elements; $\rho$: energy density, $p$: pressure (as measured in fluid rest frame).
$\rho = \rho_0(1+\epsilon)$, with $\rho_0$: rest mass density and $\epsilon$: specific internal energy (per unit mass).
Specific enthalpy: $h = 1 + \epsilon + \frac{p}{\rho_0}$, "Total energy to do work per unit mass."

This gives: $T_{\mu\nu} = \rho_0 h u_\mu u_\nu + p\ \eta_{\mu\nu}$.
It's common to write: $\rho_0 = n\ M$, with $n$: number density and $M$: rest mass of fluid particles.

Notice that $\rho$ above is not "$\rho$" we had before in the ADM decomposition:
$\rho_{ADM} = n^\mu n^\nu T_{\mu\nu} = \rho_0 h (u_\mu n^\mu)^2 - p = \rho_0 h W^2 - p$,
where we introduced $W \equiv -u^\mu n_\mu = u^0$, because in Minkowski (intertial frame) $n_\mu = (-1, \vec{0})$.
Note that $n^\mu \neq u^\mu$ in general. $u_\mu u^\mu = -1$ holds and implies $W = (1 + \sum_i (u^i)^2)^{1/2}$.
But $v^i = \frac{u^i}{u^0}$ standard 3D speed of the fluid, from which we conclude that $W = (1 - v^2)^{-1/2}$ is the Lorentz factor.

$\rho = \rho_{ADM}$ when local coordinates follow fluid elements (Lagrangian approach).
(Is this always possible?) Note: choosing $u^\mu = n^\mu$, even in Minkowski is not possible if you want $\begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}$ as the metric.

Variables: $\quad (\rho_0, \epsilon, p, v^i) \quad \rightarrow \quad$ 6 primitive variables

Equations:
$$\left.\begin{cases} \partial_\mu(\rho_0 u^\mu) = 0 & \text{Conservation of particles} \quad \rightarrow \quad 1 \text{ eq.} \\ \partial_\mu T^{\mu\nu} = 0 & \text{Conservation of energy-momentum} \quad \rightarrow \quad 4 \text{ eqs.} \end{cases}\right\} 5 \text{ equations}$$

Need one more equation: Equation of state: $p = p(\rho_0, \epsilon)$

Let's get the equations of motion:
$D = \rho_0 W$: rest mass density as seen in Eulerian frame $\Rightarrow \partial_t D + \partial_k(D\, v^k) = 0$
Continuity equation (from Conservation of particles)

Define: $S^\mu := \rho_0 h W u^\mu$; spatial comp. $S^i = \rho_0 h W^2 v^i$: momentum density in Eulerian frame.
From $T^\nu_\mu = \frac{S_\mu u^\nu}{W} + p\delta^\nu_\mu \quad \rightarrow \quad \partial_t S_i + \partial_k(S_i v^k) + \partial_i p = 0 \qquad$ Euler equations,
momentum can change because of flow of momentum "$\partial_k(\ )$" and force of pressure "$\partial_i p$".

One equation missing! Define: $\mathcal{E} = \rho_{ADM} - \rho_0 W = \rho_{ADM} - D = \rho_0 h W^2 - p - D$, the difference

between total energy density and mass energy density as measured in Eulerian frame. (This variable chosen, because it allows to find an equation in balance law form).
Notice $S^0 = \rho_0 h W^2 = \mathcal{E}+D+p$. From the conservation of energy: $0 = \partial_\mu T^{\mu 0} = \partial_\mu \left( \frac{S^0 u^\mu}{W} + p\, \eta^{\mu 0} \right)$
$\Rightarrow \quad \partial_t \mathcal{E} + \partial_k [(\mathcal{E}+p) v^k] = 0.$

Summary:

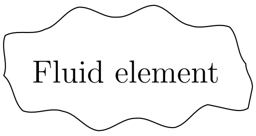
$$\text{Conserved variables:} \left\{ \begin{aligned} \partial_t D + \partial_k (D\, v^k) &= 0 \\ \partial_t S_i + \partial_k (S_i v^k) + \partial_i p &= 0 \\ \partial_t \mathcal{E} + \partial_k [(\mathcal{E}+p) v^k] &= 0 \end{aligned} \right. \qquad \text{Relation to “primitive” variables:} \left\{ \begin{aligned} D &= \rho_0 W \\ S_i &= \rho_0 h W^2 v^i \\ \mathcal{E} &= \rho_0 h W^2 - p - \rho_0 W \end{aligned} \right.$$

Valencia Formulation (1994)

A word on thermodynamics: [Still SRHD]
Consider contraction $u_\mu \partial_\nu T^{\mu\nu} = 0 \quad \Rightarrow \quad u^\mu \partial_\mu p - \rho_0 u^\mu \partial_\mu h = 0$ (used $u_\mu \partial_\nu u^\mu = 0$ from conservation of particles). But $h = 1 + \epsilon + \frac{p}{\rho_0}$

$$\Rightarrow \quad \frac{d\epsilon}{d\tau} + p \frac{d}{d\tau} \left( \frac{1}{\rho_0} \right) = 0, \quad \text{where } d/d\tau = u^\mu \partial_\mu \tag{7}$$

Local first law of thermodynmics. Why?



$$\left. \begin{array}{l} \cdot \text{ Rest mass M.} \\ \cdot \text{ Internal energy U.} \\ \cdot \text{ Volume V.} \end{array} \right\} \begin{array}{l} \rho_0 = \frac{M}{V} \quad \Rightarrow \quad dV = M\, d\left(\frac{1}{\rho_0}\right) \\ \epsilon = \frac{U}{M} \quad \Rightarrow \quad dU = M d\epsilon \end{array} \qquad (M \text{ constant})$$

First law:

$$\underbrace{dQ}_{\text{Heat loss/gain.}} = \underbrace{dU}_{\text{Change in internal energy.}} + \underbrace{p\, dV = M \left[ d\epsilon + p\, d\left( \frac{1}{\rho_0} \right) \right]}_{\text{“Mechanical” work done.}}$$

But $dQ = 0$ (perfect fluid: no vicosity, no heat conduction). → No heat conduction!
N.B. $dQ = TdS$ “Entropy preserved along flow lines” (for perfect fluid).
So (7) above is just this relation along the flow.

General relativistic hydrodynamics: Generalization easy!!

Perfect fluid stress-energy tensor: $T_{\mu\nu} = \rho_0 h u_\mu u_\nu + p g_{\mu\nu}$ (with 4-metric)
“Specific enthalpy” (again): $h = 1 + \epsilon + \frac{p}{\rho_0}$

Equations: $\left. \begin{aligned} \nabla_\mu (\rho_0 u^\mu) &= 0 \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned} \right\}$. Remember → $\nabla_\mu \xi^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\ \xi^\mu)$, so rewrite:

$$\left\{ \begin{array}{ll} \partial_\mu (\sqrt{-g}\ \rho_0 u^\mu) = 0 & \text{Conservation of particles. Still total divergence.} \\ \underbrace{\partial_\mu (\sqrt{-g}\ T^\mu_\nu)}_{\text{“Divergence-like term”.}} = \underbrace{\sqrt{-g}\ \Gamma^\alpha_{\mu\nu} T^\mu_\alpha\ ]}_{\text{“Connection from downstairs indices”.}} & \text{Conservation of energy and momentum.} \end{array} \right.$$

Now use 3+1 language: $g = -\alpha^2\gamma$, $W = -n^\mu u_\mu = \alpha u^0$ Lorentz factor.

Define: $v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}$ $\left[= \frac{1}{W} \perp^i_a u^a\right]$ "Speed of fluid as seen by Eulerian observers".

$D = \rho_0 W$: $\partial_t(\sqrt{\gamma}D) + \partial_k[\sqrt{\gamma}D(\alpha v^k - \beta^k)] = 0$ Conservation of particles.

$S^\mu = \rho_0 h W u^\mu$: $\partial_t(\sqrt{\gamma}S_i) + \partial_k\{\sqrt{\gamma}[S_i(\alpha v^k - \beta^k) + \alpha p \delta^k_i]\} = \alpha\sqrt{\gamma}\Gamma^\mu_{\nu i}T^\nu_\mu$ with $T^\mu_\nu = \frac{u^\mu S_\nu}{W} + p\delta^\mu_\nu$.

"Conservation of momentum". $\Gamma^\mu_{\nu i}$ part due to "Gravitational forces". GR Euler equations

Finally: $\mathcal{E} = \rho_0 h W^2 - p - D$: → Algebra →

$\partial_t(\sqrt{\gamma}\mathcal{E}) + \partial_k\{\sqrt{\gamma}[\mathcal{E}(\alpha v^k - \beta^k) + \alpha p v^k]\} = \alpha^2\sqrt{\gamma}(T^{0\mu}\partial_\mu \ln\alpha - \Gamma^0_{\mu\nu}T^{\mu\nu})$

Conserved: $(D, S_i, \mathcal{E})$, Primitive: $(\rho_0, \epsilon, p, v^i)$,
related by: $D = \rho_0 W$, $S_i = \rho_0 h W^2 v_i$, $\mathcal{E} = \rho_0 h W^2 - p - D$

Finally: $\rho_{ADM} = \mathcal{E} + D$, $j^i_{ADM} = S^i$, $S^{ADM}_{ij} = \rho_0 h W^2 v_i v_j + \gamma_{ij} p$

"Above is the form normally treated numerically". Here we still have stuff like $\Gamma^\mu_{\nu i}$.
Notice the the flux-balance law form, convenient because these equations have non-smooth (shock) solutions, and there are special methods for flux-balance equations to deal with that.
Let's write it in 3+1 language properly: → Algebra → We obtain:

$$\begin{aligned}
\partial_t D + D_k(\alpha D v^k) &= \alpha K D + \mathcal{L}_\beta D\\
\partial_t S^i + D_k[\alpha(S^i v^k + \gamma^{ik}p)] &= \alpha K S^i - (\mathcal{E} + D)D^i\alpha + \mathcal{L}_\beta S^i\\
\partial_t \mathcal{E} + D_k[\alpha v^k(\mathcal{E} + p)] &= (\mathcal{E} + p)[\alpha v^i v^j K_{ij} - v^i D_i\alpha] + \alpha K(\mathcal{E} + p) + \mathcal{L}_\beta \mathcal{E}
\end{aligned}$$

Notice: traded out "$\partial_t\sqrt{\gamma}$" terms to get "$\mathcal{L}_\beta$" on RHS.

Equations of state:

We have 5 equations for 6 unknowns. Need EOS: $p = p(\rho_0, \epsilon)\begin{cases}\rho_0: & \text{Rest mass energy density}\\ \epsilon: & \text{Specific internal energy}\end{cases}$

Key question: how does the EOS affect GWs?

Models: Since EOS not known, and for numerical simplicity need models / simple forms.

- Easiest choice: Dust: $p = 0$
  - Oppenheimer-Snyder collapse.
  - Non-uniform flow results in "shell crossing" singularities (simple shock formation).
  - Cannot make stars — nothing to hold them up!
- More realistic: Ideal gas EOS: $p = (\gamma - 1)\rho_0\epsilon$, with $\gamma$ the adiabatic index (not $\det\gamma_{ij}$). This follows from

$$\underset{\text{Pressure}}{p}\;\underset{\text{Volume}}{V} = \underset{\text{Number of particles}}{n}\;\underset{\text{Boltzmann constant}}{k}\;\underset{\text{Temperature}}{T} \tag{8}$$

Start with the first law: $dU(S,V) = TdS - p\,dV$ and identify $T = \left(\frac{\partial U}{\partial S}\right)_V$ and $p = -\left(\frac{\partial U}{\partial V}\right)_S$. Perform a change in derivatives in an equivalent way to a coordinate change: from $S, V$ to $T, \bar{V}$. Relations between coordinates: $\bar{V} = V,\ T = T(S,V)$ so that $\left(\frac{\partial}{\partial \bar{V}}\right)_T = \left(\frac{\partial V}{\partial \bar{V}}\right)_T \left(\frac{\partial}{\partial V}\right)_S + \left(\frac{\partial S}{\partial \bar{V}}\right)_T \left(\frac{\partial}{\partial S}\right)_V$, where we set $\left(\frac{\partial V}{\partial \bar{V}}\right)_T = 1$ from the coordinate change. Apply on $U$ and use the Maxwell relation $\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V$:
$\left(\frac{\partial U}{\partial \bar{V}}\right)_T = \left(\frac{\partial U}{\partial V}\right)_S + \left(\frac{\partial S}{\partial V}\right)_T \left(\frac{\partial U}{\partial S}\right)_V = -p + T\left(\frac{\partial p}{\partial T}\right)_V =$ (using (8)) $= -p + T\frac{nk}{V} = 0$
Thus $U = U(T)$.

Introduce the specific heats at constant volume and constant pressure respectively:

$$c_V = \frac{1}{M}\left(\frac{TdS}{dT}\right)_V = \left(\frac{dU}{dT}\right) \qquad \text{with} \qquad c_p = c_V\gamma$$

$$c_p = \frac{1}{M}\left(\frac{TdS}{dT}\right)_p \overset{\text{First Law}}{=} \frac{1}{M}\left[\frac{dU}{dT} + p\left(\frac{\partial V}{\partial T}\right)_p\right] = c_V\left(1 + \frac{nk}{Mc_V}\right)$$

So $\gamma = \left(1 + \frac{nk}{Mc_V}\right)$. If $c_V$ is constant, then $U = M\,c_V T$: $\gamma - 1 = \frac{nk}{Mc_V} = \frac{nkT}{U} = \frac{pV}{U}$. Isolating the pressure: $p = (\gamma - 1)\frac{U}{V} = (\gamma - 1)\rho_0\epsilon$.

This model can support stars and is often used.

- Polytropic EOS: $p = K\rho_0^\Gamma = K\rho_0^{1+1/N}$, with $N$: constant, polytropic index, $\Gamma$: constant, "adiabatic index of polytrope". Careful! not necessarily $\gamma$.
  Consider an adiabatic process ($dQ = 0$, no heat transfer) for the ideal gas.
  From first law: $0 = d\epsilon + p\ d\left(\frac{1}{\rho_0}\right) = \frac{1}{\gamma-1}d\left(\frac{p1}{\rho_0}\right) + p\ d\left(\frac{1}{\rho_0}\right)$, which implies $\frac{dp}{p} = \gamma\frac{d\rho_0}{\rho_0}$ → integrate to $p = K\rho_0^\gamma$ with $K$ some constant. Only in adiabatic process involving an ideal gas $\Gamma = \gamma$. However, polytrope is used even when there is heating and is a common choice in simulations. Popular modifications are "piecewise polytropic EOS", where different pieces are glued together to interpolate some desired EOS (from tables).

Astro/numerical comments:

- Basic influence of EOS on GWs? Stiffer (higher $p$) HMNS 'merger remnant' survives longer before it collapses to a BH. → Complicated waveforms! We hope that in the future this will constrain EOS by observations of GWs.
- Scale invariance gone: NS mass is $\leq 2M_\odot$.
- Numerical work is harder: shocks mean accuracy necessarily worse at same computational cost, and slow convergence. Better methods / PDE understanding desired.

### Hyperbolicity and the speed of sound

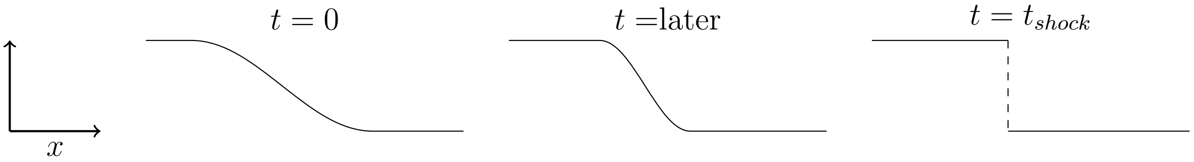
Hyperbolicity depends on EOS, but it's generally fine. Causality is used to rule out some EOSs.

Idea: write system as $\partial_\mu F^\mu(u) = s(u)$, with $F$ the fluxes and $u$ the variables. Strongly hyperbolic? Construct Jacobian matrices $A^\mu_{ij} = \frac{\partial F^\mu_i}{\partial x^j}$ and consider arbitrary vectors $\xi^\mu$ and $\zeta^\mu$ satisfying $\xi_\mu\xi^\mu = -1,\ \zeta_\mu\zeta^\mu = 1,\ \xi_\mu\zeta^\mu = 0$.

System is strongly hyperbolic if the matrix $A^\mu \xi_\mu$ is invertible (i.e., non-zero determinant) and the principal symbol $A^s = (A^\mu \xi_\mu)^{-1}(A^\mu \zeta_\mu)$ has real eigenvalues and complete set of eigenvectors [+ technical conditions].
Choose as main variables $u = (\rho_0, v^i, \epsilon)$:
fluxes along the time direction are $F^0_1 = D,\ F^0_{i+1} = S^i,\ F^0_5 = \mathcal{E}$, and along
the $x$ direction are $F^x_1 = (\alpha v^x - \beta^x)D,\ F^x_{i+1} = (\alpha v^x - \beta^x)S^i + \alpha\gamma^{xi}p,\ F^x_5 = (\alpha v^x - \beta^x)\mathcal{E} + \alpha p v^x$.
$\rightarrow$ Algebra $\rightarrow$
System is strongly hyperbolic, with 5 eigenvalues: $\lambda_0 = -\beta^x + \alpha v^x$ [multiplicity 3],
$\lambda_\pm = -\beta^x + \frac{\alpha}{1-v^2c_s^2}\left\{v^x(1-c_s^2) \pm c_s\sqrt{(1-v^2)[\gamma^{xx}(1-v^2c_s^2) - (v^x)^2(1-c_s^2)]}\right\}$, with $v^2 = \gamma^{ij}v_iv_j$.
The local speed of sound (speed at which density perturbations travel as seen in the fluid's reference frame) is defined as $c_s^2 = \frac{1}{h}\left(\chi + \frac{p}{\rho_0^2}\kappa\right)$, where $\chi = \partial p/\partial\rho_0$ and $\kappa = \partial p/\partial\epsilon$.

Weak solutions and the Riemann problem: For linear hyperbolic systems, smooth data stays smooth, so that non-smooth data will "just" propagate. Not true for non-linear systems!

Burgers equation: $\partial_t u + u\partial_x u = 0$ (strong form). An advection equation, but the speed is the solution. The wave "breaks". The solution exists only for finite time.



This happens in the Euler equations. It means that the fluid model (vanishing viscosity) breaks down. But we want to keep using them, so work with "integral form" of conservation law:
$\partial_t u + \partial_x F(u) = 0 \quad \rightarrow \quad \int_0^\infty \int_{-\infty}^\infty \phi(\partial_t u + \partial_x F(u))dx\,dt = 0 \qquad \Rightarrow$
$\int_0^\infty \int_{-\infty}^\infty (u\partial_t\phi + F\partial_x\phi)dx\,dt = -\int_{-\infty}^\infty \phi(x,0)u(x,0)dx$.
$u$ is a weak solution if the previous equations holds for $\forall\phi$. Try to understand these solutions.

Riemann problem: Example: Burgers equation with $u(x,0) = \begin{cases} u_L & x<0 \\ u_R & x>0 \end{cases}$
(1): With $u_L > u_R$: there is a unique weak solution with speed $s = (u_L + u_R)/2$. Generally speed is $s = \frac{[F]}{[u]}$, where [ ] = "jump in". Solution is a shock wave.
Rankine-Hugonot jump condition: governs conservation laws across discontinuities.

(2): With $u_L < u_R$: the weak solution is not unique. Burgers: $u(x,t) = \begin{cases} u_L & x < u_L t \\ x/t & u_L t \le x \le u_R t \\ u_R & x \ge u_R t \end{cases}$
Solution is an interpolating solution, a rarefaction wave.
How do we choose the "physical solution"? Entropy conditions. [Stable solution]

These ideas can be generalised to systems.

- Work with strong-form PDEs. Use entropy conditions to choose physical "weak" solution.
- Use numerical methods (HRSC = "high-resolution shock capturing") that naturally avoid computing derivatives across discontinuities.

Electromagnetohydrodynamics [based on Shibata's *Numerical Relativity*, sec. 4.6]:

Electric and magnetic fields $E^a, B^a$ (spatial – on $\Sigma_t$, $n_aE^a = n_aB^a = 0$) and electric current $j^a$.
Antisymmetric electromagnetic tensor: $F^{ab} = 2n^{[a}E^{b]} + \epsilon^{abc}B_c$, with $\epsilon^{abc}$ 3-Levi-Civita tensor ($= n^d\epsilon_{dabc}$). Thus $E^a = F^{ab}n_b, B^a = \frac{1}{2}\epsilon^{abc}F_{bc}$.
Current decomposed as $j^a = \rho_e n^a + \bar{j}^a$, with $\rho_e := -n_a j^a$ the electric charge density defined on $\Sigma_t$ and $\bar{j}^a = \gamma^a_b j^b$ the electric current vector on $\Sigma_t$.

Maxwell's equations: $\nabla_a F^{ab} = -4\pi j^b$, $\nabla_{[a}F_{bc]} = 0$. Different ingredients:

- Continuity equation for electric charge: $\nabla_a j^a = 0 \to$ evolution equation for $\rho_e$.
  $\partial_t(\sqrt{\gamma}\rho_e) + \partial_k(\sqrt{\gamma}[\alpha\bar{j}^k - \rho_e\beta^k]) = 0$
  Ohm's law: $j^a + (j^b u_b)u^a = \sigma_c F^{ab}u_b$, with $\sigma_c$ conductivity ($= \infty$ for ideal MHD). Take $\bar{j}^a$ and put in equation above:
  $\partial_t(\sqrt{\gamma}\rho_e) + \partial_k(\sqrt{\gamma}\rho_e v^k) = \sigma_c\partial_k\left\{\sqrt{\gamma}[(v^k + \beta^k)E^j u_j - \alpha(W\,E^k + \epsilon^{kij}u_iB_j)]\right\}$
- Constraint equations: Gauss law: $D_aE^a = 4\pi\rho_e$, in coord basis: $\partial_k(\sqrt{\gamma}E^k) = 4\pi\sqrt{\gamma}\rho_e$ and no-monopole constraint: $D_aB^a = 0$, in coord basis: $\partial_k(\sqrt{\gamma}B^k) = 0$
- Evolution equations for $E^i$ and $B^i$ (Ampère-Maxwell's law and Faraday's law):

$$\left.\begin{aligned} \partial_t E^i - \mathcal{L}_\beta E^i &= \alpha K E^i - D_k(\alpha\epsilon^{kij}B_j) - 4\pi\alpha\bar{j}^i \\ \partial_t B^i - \mathcal{L}_\beta B^i &= \alpha K B^i + D_k(\alpha\epsilon^{kij}E_j) \end{aligned}\right\}\text{3+1 language}$$

$$\left.\begin{aligned} \partial_t(\sqrt{\gamma}E^i) &= -\partial_k[\sqrt{\gamma}(2\beta^{[i}E^{k]} + \alpha\epsilon^{kij}B_j)] - 4\pi\sqrt{\gamma}(\alpha\bar{j}^i - \beta^i\rho_e) \\ \partial_t(\sqrt{\gamma}B^i) &= -\partial_k[\sqrt{\gamma}(2\beta^{[i}B^{k]} - \alpha\epsilon^{kij}E_j)] \end{aligned}\right\}\text{Conservative form}$$

Now the energy-momentum tensor has 2 parts: $T_{ab} = T^{\mathrm{HD}}_{ab} + T^{\mathrm{EM}}_{ab}$ (with $T^{\mathrm{HD}}_{ab} = \rho_0 h u_a u_b + p g_{ab}$) and $T^{\mathrm{EM}}_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b{}^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) = \frac{1}{4\pi}\left[\frac{E_aE^a + B_aB^a}{2}(\gamma_{ab} + n_an_b) - E_aE_b - B_aB_b + 2n_{(a}\epsilon_{b)cd}E^cB^d\right]$.
The 3+1 decomposition of $T_{ab}$ in the Einstein equations will now have hydrodynamical and electromagnetic terms. From above we have that $\nabla^b T^{EM}_{ab} = -F_{ab}j^b$.
Conservation of energy-momentum yields: $\nabla^b T_{ab} = 0 = \nabla^b T^{HD}_{ab} + \nabla^b T^{EM}_{ab} \;\Rightarrow\; \nabla^b T^{HD}_{ab} = F_{ab}j^b$, where the electromagnetic forces act as source. Two options:

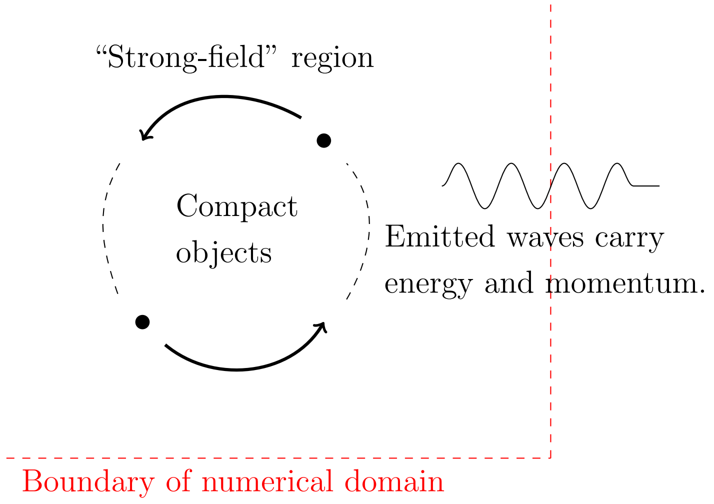
- Treat electromagnetic force as external force and write only fluid part in conservative form. Conservation of total momentum and energy not guaranteed. Shocks not accurately captured (electromagnetic speeds not taken into account in the advection terms).
- Fully conservative form. Commonly used for MHD.

Ideal magnetohydrodynamics:
Conductivity $\sigma_c = \infty \to$ require $F^{ab}u_b = 0$. This implies that the electric field in the frame comoving with fluid vanishes and $E^i = -\frac{1}{W}\epsilon^{ijk}u_jB_k$, so $E^i$ is obtained from $B^i$ – no need to evolve $E^i$! Solve the conservative form of the ideal magnetohydrodynamics equations.

MHD important for the description of jets and electromagnetic counterparts.

Not covered here: radiation transfer (Boltzmann's equation, momentum formalism, leakage scheme) $\to$ neutrinos, microphysics (electron fraction), ...

# 6 Gravitational wave extraction



Gravitational waves (GWs):

- Perturbations of spacetime travelling at the speed of light.
- Interferometers need waveforms [matched filtering].
- GWs not raw output of numerical simulations.

How can we read them off? Two main methods:

- Perturbations of
  - Schwarzschild → Regge-Wheeler-Zerilli equations.
  - Kerr → Teukolsky equation.
- Newman-Penrose formalism. More popular! We will only discuss this.

The Weyl tensor: $C_{abcd} = {}^{(n)}R_{abcd} - \frac{2}{n-2}\left(g_{a[c}{}^{(n)}R_{d]b} - g_{b[c}{}^{(n)}R_{d]a}\right) + \frac{2}{(n-1)(n-2)} g_{a[c}g_{d]b}{}^{(n)}R$

- Its expression depends on the number of spacetime dimensions $n$ (for us here $n = 4$).
- $n(n+1)(n+2)(n-3)/12 = 10$ indep. comp. → same symmetries as Riemann, and tracefree.
- Conformally invariant $\tilde{C}^a{}_{bcd} = C^a{}_{bcd}$, under $g_{ab} \quad \rightarrow \quad \tilde{g}_{ab} = \Omega^2 g_{ab}$.

Bianchi identities: $\nabla_a C^a{}_{bcd} = \nabla_{[c}R_{d]b} + \frac{1}{6}g_{b[c}\nabla_{d]}R = 8\pi\left[\nabla_{[c}T_{d]b} + \frac{1}{3}g_{b[c}\nabla_{d]}T\right]$
(same structure as the Maxwell equations)

$$\left.\begin{array}{ll} \text{Electric part:} & E_{cd} = n^a n^b\, C_{acbd} \\ \text{Magnetic part:} & B_{cd} = n^a n^b\, {}^*C_{acbd} \end{array}\right\}\ \text{with}\ \left\{\begin{array}{l} n^a \text{ arbitrary timelike unit vector and} \\ {}^*C_{abcd} = \frac{1}{2}\epsilon_{cd}{}^{ef}C_{abef} \text{ the dual Weyl tensor.} \end{array}\right.$$

Compare to electromagnetism: electric $n^b F_{ab} = E_a$ and magnetic $n^{b*}F_{ab} = B_a$ fields, with ${}^*F_{ab} = -\frac{1}{2}\epsilon_{ab}{}^{cd}F_{cd}$ and $\epsilon_{abcd}$ the 4D Levi-Civita tensor.

$E_{ab}$ and $B_{ab}$ are spacelike, $n^b E_{ab} = n^b B_{ab} = 0$, due to the symmetry of Weyl.
$E_{ab}$ and $B_{ab}$ are symmetric and tracefree → 10 independent components (like Weyl), 5 each.
$C_{abcd} = 2\left[l_{a[c}E_{d]b} - l_{b[c}E_{d]a} - n_{[c}B_{d]e}\epsilon^e{}_{ab} - n_{[a}B_{b]e}\epsilon^e{}_{cd}\right]$, with $l_{ab} = g_{ab} + 2n_a n_b$ and $\epsilon_{abc} = n^d \epsilon_{dabc}$.

3+1 language: using field equations of GR [in vacuum]

$$E_{ij} = \left[R_{ij} + KK_{ij} - K_{ik}K^k_j\right]^{TF}, \quad B_{ij} = \epsilon_{(i|}{}^{kl} D_k K_{l|j)} \tag{9}$$

$E_{ij}$ and $B_{ij}$ satisfy equations analogous to the Maxwell equations namely:
$D^j E_{ij} = B_{jk} K^j_l \epsilon^{kl}{}_i$, $D^j B_{ij} = -E_{jk} K^j_l \epsilon^{kl}{}_i$,
and propagating like Maxwell, with acceleration $a_k = D_k \ln \alpha$ and $\epsilon_{abc}$ the spatial Levi-Civita:

$$\partial_t E_{ij} = \mathcal{L}_\beta E_{ij} + \alpha[D_k B_{l(i}\epsilon_{j)}{}^{kl} - 3E^k{}_{(i}K_{j)k} + KE_{ij} - \epsilon_i{}^{kl}\epsilon_j{}^{mn}E_{km}K_{ln} + 2a_k B_{l(i}\epsilon_{j)}{}^{kl}]$$
$$\partial_t B_{ij} = \mathcal{L}_\beta B_{ij} + \alpha[-D_k E_{l(i}\epsilon_{j)}{}^{kl} - 3B^k{}_{(i}K_{j)k} + KB_{ij} - \epsilon_i{}^{kl}\epsilon_j{}^{mn}B_{km}K_{ln} - 2a_k E_{l(i}\epsilon_{j)}{}^{kl}]$$

Newman-Penrose null tetrads:
Supose we have an orthonormal tetrad $(e_A)^a$, with $A$ tetrad label and $a$ vector label.
$(e_A)^a (e_B)^b g_{ab} = \eta_{AB}$, with $\eta_{AB}$ a constant matrix $diag(-1,1,1,1)$.

$$\left.\begin{array}{rl} (e_0)^a = & \text{timelike unit vector} \\ (e_1)^a = & \text{asymptotically radialy outward unit vector} \end{array}\right\} \rightarrow \text{Build null tetrad!}$$

$$\left.\begin{array}{rl} l^a = & \frac{1}{\sqrt{2}}\left((e_0)^a + (e_1)^a\right) \\ k^a = & \frac{1}{\sqrt{2}}\left((e_0)^a - (e_1)^a\right) \\ m^a = & \frac{1}{\sqrt{2}}\left((e_2)^a + i(e_3)^a\right) \\ \bar{m}^a = & \frac{1}{\sqrt{2}}\left((e_2)^a - i(e_3)^a\right) \end{array}\right\}$$

Null tetrad: $l^a l_a = k^a k_a = m^a m_a = \bar{m}^a \bar{m}_a = 0$,
$l^a k_a = -m_a \bar{m}^a = -1$, other contractions vanish.
Note: large (6 parameters) freedom in choice of tetrad.
We can use the null tetrad as a tensor basis.
Let's use it to decompose the Weyl tensor.

The Weyl scalars:
$\Psi_0 = C_{abcd} l^a m^b l^c m^d$, $\Psi_1 = C_{abcd} l^a k^b l^c m^d$, $\Psi_2 = C_{abcd} l^a m^b \bar{m}^c k^d$, $\Psi_3 = C_{abcd} l^a k^b \bar{m}^c k^d$,
$\Psi_4 = C_{abcd} k^a \bar{m}^b k^c \bar{m}^d$ → 5 complex scalars (10 components of Weyl).
In terms of electric and magnetic parts: $Q_{ij} = E_{ij} - iB_{ij}$, $\Psi_4 = Q_{ij}\bar{m}^i \bar{m}^j$, ... , $\Psi_0 = Q_{ij} m^i m^j$.
$\Psi_4$: outgoing GWs (far from source). ($\Psi_0$: ingoing GWs).
Classic expectation: Peeling: far from isolated source: $\Psi_n \sim \frac{1}{r^{5-n}}$ (depends on notion of isolated). (Side note: the Petrov classification of spacetimes depends on relations between $\Psi_A$).

In NR we can use (9) and construct $\Psi_A$ by contraction with $l^a$, $k^a$, $m^a$, $\bar{m}^a$, which we have to build. Like $l^a = \frac{1}{\sqrt{2}}(n^a + s^a)$, for example.

Why should we bother? What physics does it tell us?

Energy and momentum of GWs:
Recap of GWs: in vacuum, linearize around flat space $h_{\mu\nu} = h^+ A^+_{\mu\nu} + h^\times A^\times_{\mu\nu}$ (+ polarization + × polarization), with $\Box h^+ = 0$, $\Box h^\times = 0$, and $A_{\mu\nu} l^\mu = 0$, $A_{\mu\nu} n^\mu = 0$, $A^\mu_\mu = 0$ and $l^a$ null and $n^a$ timelike. $h$ are the amplitudes and $A_{\mu\nu}$ the constant symmetric polarization tensors.
Considering plane waves (in TT gauge) outgoing in $r$: $h = h(r-t)$, $\partial_r h = -\partial_t h$. →

$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, $\Psi_4 = -\frac{1}{4}(\partial_t^2 h^+ - 2\partial_t\partial_r h^+ + \partial_r^2 h^+) + \frac{i}{4}(\partial_t^2 h^\times - 2\partial_t\partial_r h^\times + \partial_r^2 h^\times)$
$\Psi_4 = -\ddot{h}^+ + i\ddot{h}^\times = -\ddot{H}$ with $H = h^+ - ih^\times$. Thus $\Rightarrow$ $H = -\int_{-\infty}^{t}\int_{-\infty}^{t'} \Psi_4 dt'' dt' \approx h^+ - ih^\times$
This is the gravitational wave strain, which we calculate from $\Psi_4$ that we extract from the code. (For ingoing waves, $\partial_r h = \partial_t h$, the non-vanishing Weyls scalar is $\Psi_0$).

In a linear approximation the Isaacson stress-energy tensor in locally Cartesian coords is $T_{\mu\nu} = \frac{1}{16\pi}\langle \partial_\mu h^+ \partial_\nu h^+ + \partial_\mu h^\times \partial_\nu h^\times \rangle$, with $\langle\rangle$ "average over several wavelengths".

Energy flux:
$\frac{dE}{dtdA} = T^{0r} = \frac{1}{16\pi}\mathrm{Re}\langle \partial^0 H \partial^r \bar{H}\rangle = -\frac{1}{16\pi}\mathrm{Re}\langle \partial_t H \partial_r \bar{H}\rangle = -\frac{1}{16\pi}\langle \dot{H}\bar{H}'\rangle$, for outgoing $\partial_t h = -\partial_r h$:
$\frac{dE}{dtdA} = \frac{1}{16\pi}\langle \dot{H}\dot{\bar{H}}\rangle = \frac{1}{16\pi}\langle |\dot{H}|^2\rangle$, with $dA = r^2 d\Omega$ "area element orthogonal to radial direction".

Total flux of energy:
$\frac{dE}{dt} = \lim_{r\to\infty} \frac{r^2}{16\pi} \oint |\dot{H}|^2 d\Omega =$ total energy leaving the system $= \lim_{r\to\infty} \frac{r^2}{16\pi} \oint |\int_{-\infty}^{t} \Psi_4 dt'|^2 d\Omega$

Similarly for momentum:
$\frac{dP_i}{dtdA} = T_{ir} = \frac{1}{16\pi}\mathrm{Re}\langle \partial_i H \partial_r \bar{H}\rangle \approx \frac{1}{16\pi} l_i \langle |\dot{H}|^2\rangle$, with $l_i$ unit radial vector.
$\frac{dP_i}{dt} = \lim_{r\to\infty} \frac{r^2}{16\pi} \oint l_i |\dot{H}|^2 d\Omega = \lim_{r\to\infty} \frac{r^2}{16\pi} \oint l_i |\int_{-\infty}^{t} \Psi_4 dt'|^2 d\Omega$

... angular momentum ...

- Very often decompose $\Psi_4$ "outgoing GWs" into spherical (or spheroidal) harmonics. Relative strengths of multipoles tell us about geometry.
- Numerically only have finite $r$. Compute signal at several extraction radii and extrapolate. Or use CCE/M or hyperboloidal.

# 7 Recap class

What have we seen? Recipe we introduced at the beginning:

1. Physical problem
2. Formulation
3. PDEs analysis
4. Select numerical method
5. Implementation
6. Evaluate errors
7. Physical interpretation

1. Physical problem: (1 example) Binaries: We saw how to construct BBH ID and how to extract the GW signal from a binary. BBH, BNS

2. Formulation: We saw how to write NR as an IVP with constraints, using the 3+1 formalism.

$$\begin{cases} \partial_t \gamma_{ab} = -2\alpha K_{ab} + \mathcal{L}_\beta \gamma_{ab} \\ \partial_t K_{ab} = -D_a D_b \alpha + \alpha[R_{ab} + K K_{ab} - 2K_a{}^c K_{bc}] - 8\pi\alpha[S_{ab} - \frac{1}{2}\gamma_{ab}(S-\rho)] + \mathcal{L}_\beta K_{ab} \\ H = R + K^2 - K_{ab}K^{ab} - 16\pi\rho = 0 \\ M_a = D_b K^b{}_a - D_a K - 8\pi j_a = 0 \end{cases}$$

For BNS: need GR+Hydro equations!!

3. PDEs analysis:

- Introduced idea of hyperbolicity and thought about well-posedness of the I(B)VP. Remember: strong hyperbolicity! $\partial_t u = A^p \partial_p u + S$. $(A^p S_p)$ - principal symbol, full set of eigenvalues and eigenvectors [plus technical conditions].
- Gauge conditions.
- We also saw how to turn the constraints into an elliptic PDE.

4., 5. and 6. only really in projects.

7. Physical interpretation: (at least tools for the job).

- Event vs apparent horizons.
- Gravitational wave extraction.

Thanks for listening!! Questions?